Optimal Approximation by Blaschke Forms and Rational Functions *

Tao Qian † and Elias Wegert ‡

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Abstract We study best approximation to functions in Hardy $H^2(\mathbf{D})$ by two classes of functions of which one is n-partial fractions with poles outside the closed unit disc and the other is n-Blaschke forms. Through the equal relationship between the two classes we obtain the existence of the minimizers in both classes. The algorithm for the minimizers for small orders are practical.

Key Words: Approximation by rational functions, rational orthogonal system, Takenaka-Malmquist system, analytic signal, instantaneous frequency, adaptive decomposition, mono-components

1 Introduction

By *modified Blaschke products* (*of finite order*) we mean the functions

$$B_1(z) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-|a_1|^2}}{1-\overline{a}_1 z}, \quad B_2(z) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-|a_2|^2}}{1-\overline{a}_2 z}\frac{z-a_1}{1-\overline{a}_1 z}, \quad ...,$$

$$B_k(z) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z}\prod_{j=1}^{k-1}\frac{z-a_j}{1-\overline{a}_j z}, \; ..., \tag{1.1}$$

where the sequence $\{a_k\}_{k=1}^{\infty}$ defining the system is contained in $\mathbf{D}$, where $\mathbf{D}$ stands for the open unit disc. $\{B_k\}_{k=1}^{\infty}$ is an orthonormal system, regarded as *rational orthogonal* (or *Takenaka-Malmquist*) *system* ([1] and its references). We also use the notation

$$B_k = B_{\{a_1,...,a_k\}}$$

to indicate the dependence of B_k on the k-tuple $\{a_1, ..., a_k\}$. The system has been well studied since 1920's with ample applications in the applied mathematics, including control theory, system identification, and signal analysis. If, in particular, $a_k = 0$ for all k, then the system reduces to the Fourier basis

$$\{\frac{1}{\sqrt{2\pi}}, \frac{z}{\sqrt{2\pi}}, \frac{z^2}{\sqrt{2\pi}}, ..., \frac{z^n}{\sqrt{2\pi}}, ...\}.$$

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†Department of Mathematics, Faculty of Science and Technology, University of Macau, Taipa, Macao, China. E-mail:*fsttq@umac.mo*

‡TU Bergakademie Freiberg, Germany. E-mail:*wegert@math.tu-freiberg.de*

The Laguerre basis and the two-parameter Kautz basis are all particular cases of the general system (1.1). All the classical studies of the system rely on the condition

$$\sum_{j=1}^{\infty}(1-|a_j|)=\infty. \tag{1.2}$$

Geometrically this addresses the non-hyperbolic separability of the points a_j's. It says that under the hyperbolic distance the distribution of the points a_j's concentrates at the origin. Under the condition (1.2) the system is complete in all the Hardy spaces $H^p(\mathbf{D}), 1 \leq p < \infty$, and in the disc algebra. Conversely, completeness of the system in any of the mentioned Banach spaces implies the condition (1.2) ([1]). There are parallel theories in the region outside the unit disc, and in the upper- and lower-half complex planes.

In [15] and [14] we propose an adaptive algorithm of which the points a_k defining the system are consecutively chosen according to the function $F \in H^2(\mathbf{D})$ to be decomposed. The selections of each a_k is based on the *maximal projection principle* in spirit of greedy algorithm ([11]). The consecutive selection may not gives rise to the best approximation (See (v), (vi) below), but has a simple algorithm ([15], [14]). The formed system $\{B_k\}_{k=1}^{\infty}$ using the selected a_k's may not be a basis for the above mentioned Banach spaces, it, however, offers a fast decomposition of the given function in terms of the energy. In this paper we restrict ourselves to the finite system $\{B_k\}_{k=1}^{n}$, where n is a fixed integer. Below we will be based on the following terminology.
(i) For any n-tuple $\{a_1, ..., a_n\}$ in $\mathbf{D}$ the orthonormal system

$$B_{\{a_1\}}, B_{\{a_1,a_2\}}, ..., B_{\{a_1,...,a_n\}} \tag{1.3}$$

is called *the n-Blaschke system associated with* $\{a_1, ..., a_n\}$, or simply *an n-Blaschke system*.

(ii) For an above defined n-Blaschke system and any complex numbers $c_k, k = 1, ..., n$, with $c_n \neq 0$, the sum

$$\sum_{k=1}^{n} c_k B_{\{a_1,...,a_k\}}$$

is called *a Blaschke form associated with* $\{a_1, ..., a_n\}$. An n-Blaschke form has at most n poles that are all outside the closed unit disc.

(iii) We say that a_k *has the multiplicity* $l, 1 \leq l \leq n$, in the n-tuple $\{a_1, ..., a_n\}$, if there are totally l entries, $a_{n_1}, ..., a_{n_l}, 1 \leq n_1 < \cdots < n_l = k$, such that $a_{n_1} = \cdots = a_{n_l} = a_k$. In other words, up to the entry a_k the number a_k altogether appears l times.

(iv) We call

$$\tilde{e}_{\{a_1\}},\ \tilde{e}_{\{a_2\}},\ ...,\ \tilde{e}_{\{a_n\}},$$

the n-system associated with the n-tuple $\{a_1, ..., a_n\}$, where if $a_k \neq 0$ with multiplicity l, then

$$\tilde{e}_{\{a_k\}}(z) = \frac{1}{(1-\bar{a}_k z)^l};$$

and, if $a_k = 0$ with multiplicity l, then

$$\tilde{e}_{\{a_k\}}(z) = z^{l-1}.$$

The n-system associated with an n-tuple is usually not orthogonal. The notation $\tilde{e}_{\{a_k\}}$ here is, in fact, an abuse, as it is not only dependent of the value of a_k but also dependent of its position in the n-tuple sequence $\{a_1, ..., a_n\}$.
(v) An n-Blaschke system is called a *local n-system of F*, if the n-tuple $\{a_1, ..., a_n\}$ defining the system is chosen based on the maximal projection principle

$$\begin{aligned} &\|F_k - \langle F, B_{\{a_1,...,a_{k-1},a_k\}}\rangle B_{\{a_1,...,a_{k-1},a_k\}}\| \\ &\quad = \min\{\|F_k - \langle F, B_{\{a_1,...,a_{k-1},b\}}\rangle B_{\{a_1,...,a_{k-1},b\}}\| \ : \ b \in \mathbf{D}\}, \end{aligned} \tag{1.4}$$

$k = 1, ..., n$, where $F_1 = F$, and

$$F_k = F - \sum_{l=1}^{k-1} \langle F, B_{\{a_1,...,a_l\}}\rangle B_{\{a_1,...,a_l\}}, \qquad k > 1.$$

([15], [14]). For a given function F and an integer n, there may exist more than one local n-system. We note that if $\{a_1, ..., a_n\}$ is a local system of F and $m < n$, then $\{a_1, ..., a_m\}$ is a local m-system of F. In other words, local systems have inheriting property.
(vi) An n-Blaschke system defined by $\{a_1, ..., a_n\}$ is called a *global n-system (critical point) of F* if it satisfies

$$\begin{aligned} &\|F - \sum_{k=1}^{n} \langle F, B_{\{a_1,...,a_k\}}\rangle B_{\{a_1,...,a_k\}}\| \\ &\quad = \min\{\|F - \sum_{k=1}^{n} \langle F, B_{\{b_1,...,b_k\}}\rangle B_{\{b_1,...,b_k\}}\| \ : \ b_1, ..., b_n \in \mathbf{D}\}. \end{aligned} \tag{1.5}$$

The corresponding Blaschke form

$$\sum_{k=1}^{n} \langle F, B_{\{a_1,...,a_k\}}\rangle B_{\{a_1,...,a_k\}}$$

ia called an *n-Blaschke minimizer*, and $\{a_1, ..., a_n\}$ an *n-tuple minimizer*. A particular case is that F itself is an n-Blaschke form. There may exist more than one such minimizers, in general. Global systems do not have inheriting property.

(vii) Let $\mathcal{R}_n$ be the set of all rational functions of the form

$$R(z) = \sum_{j=0}^{r-1} c_j z^j + \sum_{l=1}^{L} \sum_{k=1}^{K_l} \frac{d_l^k}{(z - b_l)^k}, \tag{1.6}$$

where $0 \le r \le n, 1 \le K_l \le n, l = 1, ..., L, c_{r-1} \prod_{l=1}^{L} d_l^{K_l} \neq 0$, and

$$r + \sum_{l=1}^{L} K_l = n.$$

If $R \in \mathcal{R}_n$, then we call *R an n-partial fraction*. Each n-partial fraction is associated with an n-tuple $\{0, ..., b_1, ..., b_2, ..., ..., b_L, ...\} = \{a_1, ..., a_n\}$. In an n-partial fraction ordering for the involved $a_1, ..., a_n$ does not matter. The number of all distinguished orderings is $n!/(m_0! \cdots m_L!)$, where

$L+1$ is the number of all distinguished a_k's in the set, and $m_l, l = 0, ..., L$, is the number of repeating of b_l. Lemma 3.1 (§3) asserts that an n-partial fraction, under every distinguished ordering of the involved $a_1, ..., a_n$, is identical with an n-Blaschke form. The converse is obviously true due to partial sum decomposition of fractions.

(viii) For $F \in H^2($

Theorem 1.2 *If $F \in H^2(\mathsf{D})$ does not belon to any $\mathcal{R}_m, m < n$, then there exists an n-partial fraction minimizer of F.*

Below $H^2(\mathsf{D})$ will be abbreviated as H^2. The existence and algorithm for best rational approximation to H^2-functions is a classical issue. The early references include [6], [20], [8], [5], [3], [2]. Although the existence has been proved to exist ([20]), from the constructive point of view no practical algorithm for the minimizers have been established. The proofs of our theorems, in turn, provide an algorithm to obtain the two types of minimizers. While it is a subject to be further studied, at least for small orders the algorithm provided in the proofs is practical.

To note some related studies may be worthwhile. It is based on our interest in adaptive decomposition into mono-components (see below for definition) that we merge to this current study. Our study on adaptive mono-component decomposition was motivated by the engineering algorithm Empirical Mode Decomposition (EMD) ([9]), as well as some related studies on analytic signals ([10], [4]). We call a complex-valued function F in $L^p(\partial\mathsf{D}), 1 \le p \le \infty$, a *complex-mono-component* if with $F(e^{it}) = \rho(t)e^{i\mu(t)}$, there hold (i) $\rho \ge 0$; (ii) $H(\rho e^{i\mu}) = -i\rho e^{i\mu}$, where H is the Hilbert transformation of the context; and (iii) $\theta' \ge 0$. Note that phase derivative θ' has to be suitably interpreted ([13]), and if and only if $\theta' \ge 0$, then θ' is called *instantaneous frequency*. If F is a real-valued signal and its (associated) analytic signal $F + iHF$ is a complex mono-component, then F itself is called a *real-mono-component*. Being of no ambiguity we use the terminology *mono-component* for both complex- and real-mono-components ([12]). A function $F(e^{it})$ is called a *pre-complex-mono-component* if $e^{iMt}F(e^{it})$ is a complex moo-component for some $M > 0$. If F is a real-valued and if its associated analytic signal is a pre-complex-mono-component, then we call F itself a *real-pre-mono-component*. Similarly, we use the terminology *pre-mono-component* for both complex- and real-pre-mono-component ([15]). It is because of the fact that the boundary values of the system functions B_n belong to the classes of mono-components or pre-mono-components that we became interested in this system. The purpose of finding various mono-component is for adaptively decomposing signals into their intrinsic constructive mono-component atoms that, by definition, have (physically meaning full or positive) (analytic) instantaneous frequencies. The most general class for complex-valued unimodular mono-components is the boundary values of inner functions including infinite Blaschke products and singular inner functions. The proof of this fact is based on the classical Julia-Wolff-Carathéodory Theorem ([13]). This inner function result has a significant impact to the study of signals of minimum phase and all-pass filters. Finding out non-unimodular mono-components from the unimodular ones leads to a trend of recent studies on the Bedrosian identity ([17], [16], [19], [22], [21]). Starlike and p-starlike functions give rise to more general mono-components that do not rely on the unimodular types, nor on the Bedrosian identity. The modified Blaschke products studied in this paper are p-starlike functions (see [14] or [15]) that can be formulated by using the Bedrosian identity based on finite Blaschke products ([16], [19]).

2 Formula For Remaining Energy Under n-Blaschke Form

Assume that F is an H^2-function in the unit disc D whose boundary value is again denoted by F. To stress on the fact that $B_{\{a\}}$ induces the evaluation functional at the point $a \in \mathsf{D}$ (reproducing kernel) for analytic functions (as we will see), we use $e_{\{a\}}$ as an alternative notation for $B_{\{a\}}$.

To prove Theorem 1.1 the remaining energy representation under an n-Blaschke form will be needed. We will adopt the approach developed in [15] and [14]. A Technical alternative is the classical "Cristoffel-Darboux formula" ([18]). Let $F \in H^2$ and $F_1 = F$. Under the usual inner product

$$\langle F, G\rangle = \int_0^{2¼} F(e^{it})\overline{G(e^{it})}dt,$$

we have

$$\begin{aligned} F(z) = F_1(z) &= \left(F_1(z) - \langle F_1, e_{\{b_1\}}\rangle e_{\{b_1\}}(z)\right) + \langle F_1, e_{\{b_1\}}\rangle e_{\{b_1\}}(z) \\ &= F_2(z)\frac{z-b_1}{1-\bar{b}_1 z} + \langle F_1, B_{\{b_1\}}\rangle B_{\{b_1\}}(z), \end{aligned}$$

where by Cauchy's formula, we have

$$\langle F_1, e_{\{b_1\}}\rangle = \frac{\sqrt{1-|b_1|^2}}{\sqrt{2\pi}}\int_0^{2¼} \frac{F_1(e^{it})}{1-b_1 e^{-it}}dt = \sqrt{2\pi}\sqrt{1-|b_1|^2}F_1(b_1).$$

Then

$$\begin{aligned} F_2(z) &= \left(F_1(z) - \langle F_1, e_{\{b_1\}}\rangle e_{\{b_1\}}\right)\frac{1-\bar{b}_1 z}{z-b_1} \\ &= \left(F_1(z) - (1-|b_1|^2)\frac{F_1(b_1)}{1-\bar{b}_1 z}\right)\frac{1-\bar{b}_1 z}{z-b_1}. \end{aligned} \tag{2.7}$$

The first factor of F_2 in (2.7) has zero b_1, and hence $F_2 \in H^2$. We call the process from F_1 to F_2 the "sifting via b_1," or simply "a sifting". Performing sifting process up to the n times via, consecutively, $b_1, ..., b_n$, we obtain

$$\begin{aligned} F(z) &= F_{n+1}(z)\frac{z-b_1}{1-\bar{b}_1 z}\cdots\frac{z-b_n}{1-\bar{b}_n z} + \langle F_n, e_{\{b_n\}}\rangle B_{\{b_1;:::;b_n\}}(z) + \cdots \\ &\quad + \langle F_1, e_{\{b_1\}}\rangle B_{\{b_1\}}(z), \end{aligned} \tag{2.8}$$

where

$$\langle F_k, e_{\{b_k\}}\rangle B_{\{b_1;:::;b_k\}}(z) = \frac{(1-|b_k|^2)F_k(b_k)}{1-\bar{b}_k z}\frac{z-b_1}{1-\bar{b}_1 z}\cdots\frac{z-b_{k-1}}{1-\bar{b}_{k-1}z},$$

where the recursive formula for F_k is

$$F_k(z) = \left(F_{k-1}(z) - (1-|b_{k-1}|^2)\frac{F_{k-1}(b_{k-1})}{1-\bar{b}_{k-1}z}\right)\frac{1-\bar{b}_{k-1}z}{z-b_{k-1}}.$$

Denoting by

$$\tilde{F}_k(z) = F_k(z)\prod_{j=1}^{k-1}\frac{z-b_k}{1-\bar{b}_k z},$$

the usual $(k-1)$-th remainder, then there is a recursive formula for $\tilde{F}_k$:

$$\tilde{F}_k = \tilde{F}_{k-1} - \langle \tilde{F}_{k-1}, B_{\{b_1;:::;b_{k-1}\}}\rangle B_{\{b_1;:::;b_{k-1}\}}, \quad k = 2, 3, ...,$$

with $\tilde{F}_1 = F_1 = F$. Because Blaschke products are unimodular on the unit circle, we have

$$\|\tilde{F}_k\| = \|F_k\|. \tag{2.9}$$

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Noting that $\tilde{F}_k$ has zeros at $b_1, ..., b_k$, including the multiplicity, and $F - \tilde{F}_k$ is a linear combination of $B_{\{b_1,...,b_l\}}$, $l = 1, ..., k-1$, we have

$$\langle F, B_{\{b_1,...,b_k\}}\rangle = \langle \tilde{F}_k, B_{\{b_1,...,b_k\}}\rangle = \langle F_k, e_{\{b_k\}}\rangle.$$

Due to the mutual orthogonality between the B_k's and orthogonality between $\tilde{F}_{n+1}$ and all the B_k's, we have

$$\begin{aligned} \|\tilde{F}_{n+1}\|^2 &= \|F_{n+1}\|^2 \\ &= \|F\|^2 - \sum_{k=1}^{n} |\langle F_k, e_{\{b_k\}}\rangle|^2 \\ &= \|F\|^2 - 2\pi \sum_{k=1}^{n} (1 - |b_k|^2)|F_k(b_k)|^2. \end{aligned} \tag{2.10}$$

This is the remaining energy after consecutively n siftings via $b_1, ..., b_n$.

In a local n-system at k-th sifting process we choose, when $a_1, ..., a_{k-1}$ have already been fixed, $a_k \in \mathsf{D}$, so that (1.4) holds. In contrast, Theorem 1.1 concerns the case where all $a_1, ..., a_n$ are chosen at the same time so that (1.5) holds.

3 Gram-Schmidt Process in Relation to n-Blaschke System

It is well known that if $b_1, ..., b_n$ are mutually distinguished points in D, then the G-S process applied to the systems

$$\{\frac{1}{\quad}$$

In the G-S process we express F as a sum of a linear combination of $B_{\{b_1,...,b_l\}}, l = 1, .., k-1$, and a term orthogonal with all those $B_{\{b_1,...,b_l\}}$. In (2.8) taking $F = \tilde{e}_{\{b_k\}}$, we have

$$\begin{aligned} \tilde{e}_{\{b_k\}}(z) &= F_k(z)\frac{z-b_1}{1-\bar{b}_1 z}\cdots\frac{z-b_{k-1}}{1-\bar{b}_{k-1}z} + [\langle F_{k-1}, e_{\{b_{k-1}\}}\rangle B_{\{b_1,...,b_{k-1}\}}(z) + \cdots \\ &\quad +\langle F_1, e_{\{b_1\}}\rangle B_{\{b_1\}}(z)] \\ &= \tilde{F}_k(z) + S_k(z). \end{aligned} \tag{3.11}$$

Consider two possibilities. One is $b_k \neq 0$ that corresponds to $F(z) = \frac{1}{(1-\bar{b}_k z)^l}, 1 \leq l \leq k$. In the case F has pole $1/\bar{b}_k$ of multiple l. Each composing entry in the $S_k(z)$ part, if having the pole, then must be with multiplicity less or equal to $l-1$. This implies that the part

$$\tilde{F}_k(z) = F_k(z)\frac{z-b_1}{1-\bar{b}_1 z}\cdots\frac{z-b_{k-1}}{1-\bar{b}_{k-1}z}$$

must have the pole $1/\bar{b}_k$ of multiplicity l. On one hand, the Blaschke product

$$\frac{z-b_1}{1-\bar{b}_1 z}\cdots\frac{z-b_{k-1}}{1-\bar{b}_{k-1}z}$$

has the pole $1/\bar{b}_k$ of multiplicity $l-1$. On the other hand, in both sides of the equality (3.11), due to the G-S process, there are no poles other than $1/\bar{b}_1, ..., 1/\bar{b}_k$, together with the multiplicities, where some b_l can be zero. Therefore,

$$F_k(z) = \frac{c}{1-\bar{b}_k z}.$$

The L^2-normalization gives $c = \frac{\sqrt{1-|b_k|^2}}{\sqrt{2\pi}}$. Hence the normalization of $\tilde{F}_k$ is $B_{\{b_1,...,b_k\}}$, as desired. For the possibility $b_k = 0$ with multiplicity $l \geq 1$, taking $F(z) = z^{l-1}$, that has pole at infinity of multiplicity $l-1$, then (3.11) reads

$$\begin{aligned} z^{l-1} &= F_k(z)\prod_{l=1}^{k-1}\frac{z-b_l}{1-\bar{b}_l z} + d_{k-1}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-|b_{k-1}|^2}}{1-\bar{b}_{k-1}z}\prod_{l=1}^{k-2}\frac{z-b_l}{1-\bar{b}_l z} + \cdots \\ &= F_k(z)\left(\prod_{l=1}^{k-1}\frac{z-b_l}{1-\bar{a}_l z}\right) + S_k(z) \\ &= \tilde{F}_k(z) + S_k(z). \end{aligned} \tag{3.12}$$

We are to show that $F_k(z)$ is a constant, and so the L^2 normalization of $\tilde{F}_k$ is just $B_{\{b_1,...,b_k\}}$. We note that the Blaschke product in the round brackets in (3.12) has the order z^{l-1} at infinity. In the case if $b_{k-1} = 0$, then in each of the modified Blaschke products in $S_k(z)$ the pole $z = \infty$ is of multiplicity at most $l-3$. Dividing z^{l-1} from both sides and letting $z \to \infty$, taking into account the fact the G-S process produces no poles other than those arising from the round bracket in (3.12), we conclude $F_k = c$. If $b_{k-1} \neq 0$, then $z = \infty$ is a pole in S_k with multiplicity at most $l-2$. The same reasoning concludes $F_n = c$. The proof is thus complete.

4 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 We first observe that for any n-tuple $\{b_1, ..., b_n\}$ in D, although the system $\{B_{\{b_1,...,b_k\}}\}_{k=1}^n$ is relevant to the ordering of the b_k's in the n-tuple, the projection

$$\sum_{k=1}^{n} \langle F, B_{\{b_1,...,b_k\}}\rangle B_{\{b_1,...,b_k\}}$$

is irrelevant to it. This is because, in accordance with Lemma 3.1, it is the orthogonal projection to the complex n-dimensional linear space spanned by $\{e_{\{b_1\}}, ..., e_{\{b_n\}}\}$, where the latter is irrelevant to the order. Now we show that an n-tuple $\{a_1, ..., a_n\}$ exists in which every a_k is an interior point of D that gives rise to an n-Blaschke minimizer (see (vi) in §1). We show this by introducing a contradiction. Suppose that F itself is not an n-Blaschke form, and there does not exist a minimizer n-tuple $\{a_1, ..., a_n\}$ inside the disc. In the case there exists a sequence of n-tuples $\{b_1^l, ..., b_n^l\}$ such that

$$\|\tilde{F}_n^l\|^2 = \|F\|^2 - 2\pi \sum_{k=1}^{n} (1 - |b_k^l|^2)|F_k(b_k^l)|^2$$

tends to the infimum value d as $l \to \infty$, while for at least one index k_0 there exists a subsequence $\{b_{k_0}^{l_j}\}_{j=1}^{\infty}$ converging to a boundary point. We divide the indices $k = 1, ..., n$ into two groups, denoted by the letters $\mathcal{B}$ and $\mathcal{I}$, respectively, where if $\{b_k^l\}_{l=1}^{\infty}$ contains a subsequence converging to a boundary point of D, then $k \in \mathcal{B}$, and otherwise $k \in \mathcal{I}$. Note that $k_0 \in \mathcal{B} \neq \emptyset$. Due to the observation made at the beginning of the proof, we may alter the ordering of $\{b_1^l, ..., b_n^l\}$, if necessary, and may assume that the indices in $\mathcal{I}$ are all smaller than those in $\mathcal{B}$. By a diagonal process we can choose a subsequence $l_j \to \infty$ such that for $k \in \mathcal{B}$ the sequences $\{b_k^{l_j}\}$ converge to boundary points, and for $k \in \mathcal{I}$, converge to interior points. Without loss of generality we may assume that the original sequence $\{l\}_{l=1}^{\infty}$ has such property. That is, as $l \to \infty$, the sequence $\{b_k^l\}_{l=1}^{\infty}$ converges to $b_k \in \partial\mathsf{D}$ if $k \in \mathcal{B}$, and converges to $b_k \in \mathsf{D}$ if $k \in \mathcal{I}$. For any l adopt the notation for the non-zero remainders

$$\tilde{F}_m^l = F - \sum_{k=1}^{m} \langle F_j^l, e_{\{b_j^l\}}\rangle B_{\{b_1^l,...,b_j^l\}}, \quad m = 1, ..., n,$$

and the notation

$$R_{\mathcal{I}}^l = F - \sum_{j\in\mathcal{I}} \langle F_j^l, e_{\{b_j^l\}}\rangle B_{\{b_1^l,...,b_j^l\}}.$$

There follows, in accordance with (2.10) and (2.9),

$$\|R_{\mathcal{I}}^l\|^2 = \|F\|^2 - \sum_{j\in\mathcal{I}} |\langle F_j^l, e_{\{b_j^l\}}\rangle|^2 \geq \|\tilde{F}_k^l\|^2 = \|F_k^l\|^2, \quad k \in \mathcal{B}. \tag{4.13}$$

We have, owing the properties of the Poisson kernel P_r and the inequality (4.13), for any given $\epsilon > 0$,

$$
\begin{aligned}
\|R^l_{\mathcal{I}}\| &\geq \|R^l_{\mathcal{I}} - \sum_{k\in\mathcal{B}} \langle F^l_k, e_{\{b^l_k\}}\rangle B_{\{b^l_1;\dots;b^l_k\}}\| \\
&\geq \|P_r * (R^l_{\mathcal{I}} - \sum_{k\in\mathcal{B}} \langle F^l_k, e_{\{b^l_k\}}\rangle B_{\{b^l_1;\dots;b^l_k\}})\| \\
&\geq \|P_r * R^l_{\mathcal{I}}\| - \sum_{k\in\mathcal{B}} \|F^l_k\| \|P_r * B_{\{b^l_1;\dots;b^l_k\}}\| \\
&\geq \|P_r * R^l_{\mathcal{I}}\| - \sum_{k\in\mathcal{B}} \|R^l_{\mathcal{I}}\| \|P_r * B_{\{b^l_1;\dots;b^l_k\}}\|
\end{aligned}
$$

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$\geq (1 -$ ϵ r and the inequality $*$ [illegible] $k l$ $k\in\mathcal{B}$ F2 10190.996 -167.557 l 26

Note that $d > 0$, for otherwise we would have that F is an m-Blaschke form for some $m < n$, being contrary to the assumption. This last relation (4.14), however, shows that the selections of b_k^l's and the limit procedure involving $k \in \mathcal{B}$ all have no effect and the minimum value d can be attained at an m-tuple inside the unit disc, where $m < n$. This is a contradiction. For in that case we could select an arbitrary $\tilde{b}_k \in \mathbf{D}$ for each $k \in \mathcal{B}$. Those, together with the original limit points $b_k \in \mathbf{D}, k \in \mathcal{I}$, would give rise to an n-Blaschke system definitely improving the approximation given in (4.14). This proves the Theorem.

Proof of Theorem 1.2 It is known that any $R \in \mathcal{R}_n$ is an n-Blaschke form. On the other hand, every n-Blaschke form is an $R \in \mathcal{R}_n$. Now, by the assumption, F itself is not in any $\mathcal{R}_m$ for $m < n$, and F, therefore, is not an m-Blaschke form for $m < n$. By using Theorem 1.1, F has an n-Blaschke minimizer, that is a function in $\mathcal{R}_n$. We claim that it is a minimizer of F in $\mathcal{R}_n$. For, if this were not true, then there would exist another rational function in $\mathcal{R}_n$ offering a better approximation to F. It is also an n-Blaschke and thus presents a contradiction. The proof is complete.

Remark The proof of Theorem 1.1 and the relations obtained in §2 offerd an algorithm for an n-tuple minimizer. For small n the algorithm is practical.

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