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Orthonormal bases with nonlinear phases

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Abstract For adaptive representation of nonlinear signals, the bank M of real square integrable functions that have nonlinear phases and nonnegative instantaneous frequencies under the anal tic signal method is investigated. A particular class of functions with e plicit e pressions in M is obtained using

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recent results on the Bedrosian identit . We then construct orthonormal bases for the Hilbert space of real square integrable functions with the basis functions from M.

Keywords The Hilbert transform · The empirical mode decomposition · Time-frequenc anal sis · Orthonormal bases · Hard spaces

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1 Introduction

The classical Fourier basis has been proven to be an ef cient approach to represent a linear and stationar signal. However, it is not ef cient to represent a nonlinear and nonstationar signal (cf., [11]). There were man methods ([4, 6, 11, 13 15, 19] and the references therein) proposed recentl to overcome the shortcoming of the Fourier anal sis from both empirical and mathematical points of view. In particular, the stud of the instantaneous amplitude and frequenc [11] of a nonlinear and nonstationar signal has attracted much attention. A common approach to de ne the instantaneous amplitude and frequenc is through the Hilbert transform. It is desirable to construct orthonormal bases for the Hilbert space of real square integrable functions which have nonlinear phases and admit well-behaved Hilbert transforms. The goal of this paper is to present general constructions of such orthonormal bases.

We begin with recalling the de nition of the Hilbert transform. The *Fourier transform* $\hat{f}$ of a function $f \in L^1( )$ is de ned b

$$\hat{f}(\xi) = (F f)(\xi) := \int_{R} f(x) e^{-ix\xi} \mathrm{d}x, \quad \xi \in \quad .$$

We can e tend the Fourier transform to a *unitary operator* ([7], page 20) on $L^2( )$ through a standard appro imation process [10]. The *Hilbert transform* H is a bounded operator on $L^2( )$ de ned via the *Fourier multiplier* $-i\,\mathrm{sgn}$, where $\mathrm{sgn}(\xi)$ takes values -1, 0 and 1 for $\xi < 0$, $\xi = 0$ and $\xi > 0$, respectivel (cf., [10]). Speci call , we have for each $f \in L^2( )$ that

$$(Hf)\hat{}(\xi) = -i\,\mathrm{sgn}\,(\xi)\,\hat{f}(\xi), \quad \xi \in \quad . \tag{1.1}$$

Set $L^2_{\mathrm{r}}( ) := \{f : f \in L^2( ),\ f \text{ is real-valued}\}$, which is a Hilbert space over . The *analytic signal* Af of a function $f \in L^2_{\mathrm{r}}( )$ is de ned b

$$Af := f + iHf.$$

It ma be rewritten as

$$(Af)(t) = \gamma(t) e^{i\varphi(t)}, \quad t \in \tag{1.2}$$

with $\gamma \geq 0$ and φ a real function. The above equation gives f an amplitude-frequenc modulation

$$f(t) = \gamma(t) \cos \varphi(t), \quad t \in \quad .$$

The values $\gamma(t)$ and $\varphi(t)$ above are then considered as the instantaneous amplitude and phase of signal f at time t, respectivel . Note that the derivative φ' taken as the instantaneous frequenc of f is ph sicall meaningful onl if it is nonnegative. We sa that f *admits a well-behaved Hilbert transform* if the derivative of φ in (1.2) is nonnegative.

In general, a real signal ma not admit a well-behaved Hilbert transform. The *empirical mode decomposition* (EMD) introduced in [11] is a numerical algorithm aiming at decomposing a signal into a sum of signals each of which admits a well-behaved Hilbert transform. The anal tic signal method can then be applied to each summand to ield a sound energ -frequenc -time distribution. The EMD algorithm works well for man applications but at the same time it does not for some cases. It is desirable to build a solid mathematical base for the algorithm. There are two stages in building such a base for the useful algorithm, [21]. The rst is to construct a large bank M of functions $f \in L_r^2(\ \ )$ such that

$$(Af)(t) = \gamma(t) e^{i\varphi(t)}, \ \gamma(t) \geq 0, \ \varphi'(t) \geq 0, \quad t \in \quad . \tag{1.3}$$

The second is to establish an adaptive and rapid algorithm to decompose an arbitrar function $f \in L_{\mathrm{r}}^2(\ \ )$ into a sum of functions in M with the summand deca ing fast.

This paper serves as a rst attempt to the above two stages. We simpl aim at enlarging the e isting class of functions with e plicit e pressions in M, and providing a wa of decomposing a square integrable function into a sum of functions in M. We shall not emphasi e the ph sical meaning of such a decomposition. Neither shall we discuss the deca ness of components of the decomposition, which, however, deserves careful attention in the future. We intend to address the issue in another occasion.

The e position of the paper is organi ed as follows. We rst construct in Section 2 a class of functions with e plicit e pressions in M using recent developments [20, 23] on the Bedrosian identit [2]. To decompose a function $f \in L_{\mathrm{r}}^2(\ \ )$ into a sum of functions in M, we construct orthonormal bases for the real Hilbert space $L_{\mathrm{r}}^2(\ \ )$ with the basis functions in M. Two constructions along with concrete e amples are presented in Sections 3 and 4. Finall in Section 5, we give similar constructions of orthonormal bases for $L_{\mathrm{r}}^2[-\pi, \pi]$, the Hilbert space of real functions in $L^2[-\pi, \pi]$.

2 Functions admitting a well-behaved Hilbert transform

We construct in this section functions that admit a well-behaved Hilbert transform. An approach for constructing functions in M was proposed b the third author in 2002. It is to find nonnegative $\gamma \in L^2(\mathbb{R})$ and real $\varphi \in C^1(\mathbb{R})$ satisfying the nonlinear singular integral equation

$$[H(\gamma(\cdot)\cos\varphi(\cdot))](t) = \gamma(t)\sin\varphi(t), \quad t \in \mathbb{R} \tag{2.1}$$

subjected to the constraint that

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \geq 0, \quad t \in \mathbb{R}. \tag{2.2}$$

Motivated by this approach, we shall first consider an equation similar to (2.1) for periodic functions and then obtain functions in M through the Cayley transform. Let us make preparations for this by introducing the Hardy spaces [8, 9, 18].

Set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$. The set of all the holomorphic functions on $\mathbb{C}_+$ and $\mathbb{D}$ is denoted by $\mathbf{H}(\mathbb{C}_+)$ and $\mathbf{H}(\mathbb{D})$, respectively. We shall work on the Hardy spaces

$$\mathbf{H}^2(\mathbb{C}_+) := \left\{h \in \mathbf{H}(\mathbb{C}_+) : \sup\left\{\int_{\mathrm{R}} |h(x+iy)|^2 \mathrm{d}x : y > 0\right\} < \infty\right\},$$

$$\mathbf{H}^2(\mathbb{D}) := \left\{h \in \mathbf{H}(\mathbb{D}) : \sup\left\{\int_{-\pi}^{\pi} |h(re^{it})|^2 \mathrm{d}t : r \in (0,1)\right\} < \infty\right\},$$

$$\mathbf{H}^\infty(\mathbb{C}_+) := \{h \in \mathbf{H}(\mathbb{C}_+) : \sup\{|h(z)| : z \in \mathbb{C}_+\} < \infty\},$$

$$\mathbf{H}^\infty(\mathbb{D}) := \{h \in \mathbf{H}(\mathbb{D}) : \sup\{|h(z)| : z \in \mathbb{D}\} < \infty\}.$$

For a fixed $\alpha > 0$, we introduce for each $t \in \mathbb{R}$ the cone in $\mathbb{C}_+$

$$\Gamma_\alpha(t) := \{z = x + iy \in \mathbb{C}_+ : |x - t| < \alpha y\}.$$

For each $f \in \mathbf{H}^p(\mathbb{C}_+)$, $p \in \{2, \infty\}$, there exists a $g \in L^p(\mathbb{R})$ such that for almost every $t \in \mathbb{R}$ there holds

$$\lim_{\Gamma_\alpha(t) \ni z \to t} f(z) = g(t).$$

Likewise, if we set for a fixed $\varepsilon \in (0,1)$ and for every $\theta \in [-\pi, \pi]$

$$\Gamma_\varepsilon(e^{i\theta}) := \{\lambda e^{i\theta} + (1-\lambda)z : \lambda \in (0,1),\ |z| < \varepsilon\}$$

then there exists for each $f \in \mathbf{H}^p(\mathbb{D})$, $p \in \{2, \infty\}$, a $g \in L^p(\partial\mathbb{D})$ such that for almost every $\theta \in [-\pi, \pi]$

$$\lim_{\Gamma_\varepsilon(e^{i\theta}) \ni z \to e^{i\theta}} f(z) = g(e^{i\theta}).$$

In both cases, we call the function g the *nontangential boundary limit* of f, which is independent of the choice of the $\alpha > 0$ or $\varepsilon \in (0,1)$. For simplicity, we shall use the same notation for a function in Hardy spaces as that for its nontangential boundar limit. The spaces $\mathbf{H}^2(\mathbb{C}_+)$ and $\mathbf{H}^2(\mathbb{U})$ are Hilbert spaces endowed, respectivel , with the inner products

$$\langle f, g\rangle_{\mathbf{H}^2(\mathbb{C}_+)} := \int_{\mathbb{R}} f(t)\overline{g(t)}\mathrm{d}t, \quad f, g \in \mathbf{H}^2(\mathbb{C}_+)$$

and

$$\langle f, g\rangle_{\mathbf{H}^2(\mathbb{U})} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}\mathrm{d}t, \quad f, g \in \mathbf{H}^2(\mathbb{U}).$$

These two spaces are connected through the *Cayley transform*. The conformal mapping from $\mathbb{C}_+$ to $\mathbb{U}$ de ned b

$$K(w) := \frac{i-w}{i+w}, \quad w \in \mathbb{C}_+ \tag{2.3}$$

is called the Ca le transform and it e tends continuousl as a bijective mapping from the e tended real line to $\partial\mathbb{U}$. The correspondence between the boundaries is

$$e^{is} = K(t) = \frac{i-t}{i+t}, \quad t \in \mathbb{R}, \ s \in (-\pi, \pi),$$

which implies

$$s = 2\arctan t, \quad t \in \mathbb{R}. \tag{2.4}$$

With the Ca le transform, the linear transformation T from $\mathbf{H}^2(\mathbb{U})$ to $\mathbf{H}^2(\mathbb{C}_+)$ de ned for $f \in \mathbf{H}^2(\mathbb{U})$ b

$$Tf := \frac{1}{\sqrt{\pi}}\frac{1}{1-iz}(f \circ K) \tag{2.5}$$

is an *isomorphism* ([7], page 19).

The Hilbert transform, as indicated b the anal tic signal approach described in the introduction, is fundamental in the time-frequenc anal sis of square integrable signals. Its counterpart in the time-frequenc anal sis of periodic signals is the *circular Hilbert transform* $\tilde{H}$ de ned for $f \in L^2[-\pi, \pi]$ b

$$(\tilde{H}f)(t) := \sum_{k\in\mathbb{Z}} -i\,\mathrm{sgn}\,(k)c_k(f)e^{ikt}, \quad t \in [-\pi, \pi], \tag{2.6}$$

where $c_k(f)$ is the kth Fourier coef cient of f de ned b

$$c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\mathrm{d}t. \tag{2.7}$$

We wish to construct functions with e plicit e pressions in the bank $\tilde{M}$ of functions $f \in L^2_{\mathrm{r}}[-\pi, \pi]$ such that

$$(f + i\tilde{H}f)(t) = \rho(t)e^{i\theta(t)}, \ \rho(t) \geq 0, \ \theta'(t) \geq 0, \quad t \in [-\pi, \pi]$$

and orthonormal bases from $\tilde{M}$ for $L^2_{\mathrm{r}}[-\pi, \pi]$. The latter question is postponed until Section 5. Functions in $\tilde{M}$ ma be obtained b nding nonnegative $\rho \in L^2[-\pi, \pi]$ and real $\theta \in C^1[-\pi, \pi]$ with a nonnegative derivative such that

$$[\tilde{H}(\rho(\cdot)\cos\theta(\cdot))](t) = \rho(t)\sin\theta(t), \quad t \in [-\pi, \pi]. \tag{2.8}$$

This section is devoted to constructing functions in M and $\tilde{M}$ with e plicit e pressions. The construction method is to solve (2.1), (2.8) with a prescribed phase function.

Let . be the set of all the positive integers and $_+ := . \cup \{0\}$. To enumerate nite sets, we de ne for each $n \in .$, $._n := \{1, 2, \ldots, n\}$ and $_n := \{0, 1, \ldots, n-1\}$

B a necessar and suf cient condition for circular Bedrosian identities established in [23], (2.12) holds if and onl if the following three equations are satis ed

$$\sum_{j\in \mathrm{N}} c_j(\rho)c_{-j}(g) = \sum_{j\in \mathrm{N}} c_{-j}(\rho)c_j(g), \tag{2.13}$$

$$c_k(\rho)c_0(g) + 2\sum_{j\in \mathrm{N}} c_{k+j}(\rho)c_{-j}(g) = 0, \quad k \in . \ , \tag{2.14}$$

$$c_{-k}(\rho)c_0(g) + 2\sum_{j\in \mathrm{N}} c_{-k-j}(\rho)c_j(g) = 0, \quad k \in . \ . \tag{2.15}$$

Noting that ρ is real and $\alpha_j \in \ $, $j \in \ _+$, we have that

$$c_{-j}(\rho) = \overline{c_j(\rho)}, \ c_0(g) = \alpha_0, \ c_j(g) = c_{-j}(g) = \frac{1}{2}c_j(e^{i\theta}) = \frac{1}{2}\alpha_j, \quad j \in . \ .$$

The above relations impl that (2.13), (2.14) and (2.15) are of the forms

$$\sum_{j\in \mathrm{N}} (c_j(\rho) - c_{-j}(\rho))\alpha_j = 0, \tag{2.16}$$

$$c_k(\rho)\alpha_0 + \sum_{j\in \mathrm{N}} c_{k+j}(\rho)\alpha_j = 0, \quad k \in . \ , \tag{2.17}$$

$$c_{-k}(\rho)\alpha_0 + \sum_{j\in \mathrm{N}} c_{-k-j}(\rho)\alpha_j = 0, \quad k \in . \ , \tag{2.18}$$

where (2.17) and (2.18) are equivalent. Thus, we get that $\rho \in L^2[-\pi,\pi]$ satis es (2.8) if and onl if there holds (2.16) and (2.17).

B the *Parseval identity* for $L^2[-\pi,\pi]$, (2.17) is equivalent to that

$$\int_{-\pi}^{\pi} \left(\sum_{j\in \mathrm{Z}_+} c_{j+1}(\rho)e^{ijt}\right) \overline{e^{ikt}e^{i\theta(t)}}\mathrm{d}t = 0, \quad k \in \ _+,$$

that is,

$$\int_{-\pi}^{\pi} \left(\sum_{j\in \mathrm{Z}_+} c_{j+1}(\rho)e^{ijt}\right) e^{-ikt} \prod_{j\in \mathrm{N}_n} \frac{1-\lambda_j e^{it}}{e^{it}-\lambda_j}\mathrm{d}t = 0, \quad k \in \ _+. \tag{2.19}$$

Equation 2.19 holds if and onl if there e ists $[c_j : j \in . \ ] \in \ ^2(\ )$ such that

$$\left(\sum_{j\in \mathrm{Z}_+} c_{j+1}(\rho)e^{ijt}\right) \prod_{j\in \mathrm{N}_n} (1-\lambda_j e^{it}) = \left(\sum_{j\in \mathrm{N}} c_j e^{-ijt}\right) \prod_{j\in \mathrm{N}_n} (e^{it}-\lambda_j), \quad t \in [-\pi,\pi]. \tag{2.20}$$

The highest order of the trigonometric series on the right hand side of (2.20) is $n-1$, while the lowest order of the trigonometric series on the left is 0. Consequentl , the trigonometric function series e pressed b both sides must have orders 0 to $n-1$ onl . Therefore, if (2.20) holds then there e ists $b_j \in$, $j \in$ $_n$, such that

$$\left(\sum_{j\in \mathrm{Z}_+} c_{j+1}(\rho)e^{ijt}\right)\prod_{j\in \mathrm{N}_n}(1-\lambda_j e^{it}) = \sum_{j\in \mathrm{Z}_n} b_j e^{ijt}, \quad t\in[-\pi,\pi]. \tag{2.21}$$

Conversel , if (2.21) holds for some $b_j \in$, $j \in$ $_n$ then (2.20) is true for some $[c_j : j \in$ $] \in$ $^2($ $)$. We hence conclude that (2.17) holds if and onl if there e ists $b_j \in$, $j \in$ $_n$, such that (2.21) holds.

Suppose that $\rho \in L^2_{\mathrm{r}}[-\pi,\pi]$ satis es (2.8). B the discussions above, there holds (2.16) and (2.21) for some $b_j \in$, $j \in$ $_n$. Since ρ is real, we obtain from (2.21) that

$$\rho(t) = 2\,\mathrm{Re}\left(\frac{e^{it}\sum_{j\in \mathrm{Z}_n} b_j e^{ijt}}{\prod_{j\in \mathrm{N}_n}(1-\lambda_j e^{it})}\right) + c_0(\rho), \quad t\in[-\pi,\pi].$$

Using the above e pression of ρ, we conclude that (2.16) is of the form

$$\int_{-\pi}^{\pi} \frac{\sum_{j\in \mathrm{Z}_n} \mathrm{Im}\,(b_j)e^{i(n-j-1)t}}{\prod_{j\in \mathrm{N}_n}(1-\lambda_j e^{it})}\mathrm{d}t = 0.$$

Through a change of variables, the above equation can be written as

$$\int_{\partial \mathrm{U}} \frac{\sum_{j\in \mathrm{Z}_n} \mathrm{Im}\,(b_j) z^{n-1-j}}{z\prod_{j\in \mathrm{N}_n}(1-\lambda_j z)} dz = 0. \tag{2.22}$$

B the Cauch integral formula (2.10), we have that $\mathrm{Im}\,(b_{n-1}) = 0$.

On the other hand, if ρ has the form (2.11) for $b_j \in$, $j \in$ $_{n-1}$ and $b_{n-1}, c \in$, then we have that

$$\sum_{j\in \mathrm{Z}_+} c_{j+1}(\rho)e^{ijt} = \frac{\frac{1}{2}\sum_{j\in \mathrm{Z}_n} b_j e^{ijt}}{\prod_{j\in \mathrm{N}_n}(1-\lambda_j e^{it})}, \quad t\in[-\pi,\pi], \tag{2.23}$$

which implies b the equivalence of (2.21) and (2.17) that (2.17) is satis ed. Equation 2.22 then follows from the Cauch integral formula and b_{n-1} being real. Together with (2.23) and the Parseval identit for $L^2[-\pi,\pi]$, (2.22) leads to (2.16). Consequentl , we conclude that ρ satis es (2.8). □

One can alwa s choose big enough c in (2.11) so that ρ given there is nonnegative. We conclude in this case that $\rho\cos\theta$ for ρ and θ so chosen is contained in $\tilde{M}$.

We ne t construct functions in M with e plicit e pressions. In this case, we choose φ to satisf the equation

$$e^{i\varphi(t)} = \frac{1+it}{\sqrt{1+t^2}} \prod_{j\in \mathrm{N}_n} \frac{e^{i2\arctan t} - \lambda_j}{1-\lambda_j e^{i2\arctan t}}, \quad t \in \quad , \tag{2.24}$$

where $n \in .$ and $\lambda_j \in [0, 1)$, $j \in ._n$. It can be veri ed directl that φ de ned above satis es (2.2). Our ne t task is to construct γ that satis es (2.1) given φ in (2.24). Along this line, we need the following characteri ation of nontangential boundar limits of functions in $\mathbf{H}^2(\ _+)$ (see, for e ample, [9], page 88 and [16]).

Lemma 2.2 *Functions $f, g \in L^2_r(\ )$ satisfies the equation $Hf = g$ if and only if $f + ig$ is the nontangential boundary limit of some function in $\mathbf{H}^2(\ _+)$.*

A similar result holds for the space $\mathbf{H}^2(\ )$ (see, for e ample, [9, 18]).

Lemma 2.3 *Let $f \in L^2[-\pi, \pi]$. Then there exists a nontangential boundary limit h of some function in $\mathbf{H}^2(\ )$ such that $f = h(e^{i\cdot})$ if and only if $c_{-j}(f) = 0$ for all $j \in .$.*

We are now read to present a construction of γ.

Theorem 2.4 *Let $n \in .$ and φ be given by (2.24) with $\lambda_j \in [0, 1)$, $j \in ._n$. Then a real function $\gamma \in L^2(\ )$ satisfies (2.1) if and only if there exists $b_j \in \ $, $j \in ._n$ and $c \in \ $ such that*

$$\gamma(t) = \frac{1}{\sqrt{1+t^2}} \left(\mathrm{Re} \left(\frac{\sum_{j\in \mathrm{N}_n} b_j e^{i2j\arctan t}}{\prod_{j\in \mathrm{N}_n} (1-\lambda_j e^{i2\arctan t})} \right) + c \right), \quad t \in \quad . \tag{2.25}$$

Proof Let φ be a real function on as described in the assumption. We set for each function γ on

$$\rho := \left(\sqrt{1+t^2}\,\gamma\right) \circ \tan\left(\frac{\cdot}{2}\right). \tag{2.26}$$

One can see that $\gamma \in L^2(\ )$ if and onl if $\rho \in L^2[-\pi, \pi]$. Note also that there holds

$$[(\gamma e^{i\varphi}(1-it)) \circ K^{-1}](e^{is}) = \rho(s)\, e^{i\theta(s)}, \quad s \in (-\pi, \pi), \tag{2.27}$$

where θ is de ned b (2.9).

Let $\gamma \in L^2_{\mathrm{r}}(\ )$. Denote b $B(\ _+)$ and $B(\ )$ the set of nontangential boundar limits of functions in $\mathbf{H}^2(\ _+)$ and $\mathbf{H}^2(\ )$, respectivel . We claim that $\gamma e^{i\varphi} \in B(\ _+)$ if and onl if there e ists an $h \in B(\ )$ such that

$$h(e^{i\cdot}) = \rho e^{i\theta}. \tag{2.28}$$

Firstl , if $\gamma e^{i\varphi} \in B(\ _+)$ then b the isomorphism (2.5), $h := (\gamma e^{i\varphi}(1-it)) \circ K^{-1} \in B(\ )$ and it satis es (2.28) b (2.27). On the other hand, suppose we have an $h \in B(\mathbb{D})$ that satisfies (2.28). Still by the isomorphism (2.5), $\frac{h \circ K}{1-it} \in B(\mathbb{C}_+)$. We obtain by (2.27) and (2.28) that

$$\gamma(t)e^{i\varphi(t)} = \frac{(h\circ K)(t)}{1-it}, \quad t \in \mathbb{R},$$

which proves that $\gamma e^{i\varphi}$ is contained in $B(\mathbb{C}_+)$.

It follows by the above claim, Lemmas 2.2 and 2.3 that γ satisfies (2.1) if and only if there exists $[\alpha_j : j \in \mathbb{Z}_+] \in l^2(\mathbb{Z}_+)$ such that

$$\rho(s)e^{i\theta(s)} = \left(\sum_{j\in \mathrm{Z}} c_j(\rho)e^{ijs}\right)\prod_{j\in \mathrm{N}_n}\frac{e^{is}-\lambda_j}{1-\lambda_j e^{is}} = \sum_{j\in \mathrm{Z}_+}\alpha_j e^{ijs}, \quad s\in[-\pi,\pi]. \tag{2.29}$$

By the definition (2.26), γ is of the form (2.25) if and only if ρ has the following form

$$\rho(s) = \operatorname{Re}\left(\frac{\sum_{j\in \mathrm{N}_n} b_j e^{ijs}}{\prod_{j\in\mathrm{N}_n}(1-\lambda_j e^{is})}\right) + c, \quad s\in[-\pi,\pi]. \tag{2.30}$$

We conclude that to prove the theorem it suffices to prove that (2.29) holds for some $[\alpha_j : j\in\mathbb{Z}_+] \in l^2(\mathbb{Z}_+)$ if and only if (2.30) holds for some $b_j \in \mathbb{C}$, $j\in\mathbb{N}_n$ and $c\in\mathbb{R}$.

Suppose first that (2.30) holds for some $b_j\in\mathbb{C}$, $j\in\mathbb{N}_n$ and $c\in\mathbb{R}$. Then through a direct computation, we get that

$$\rho(s)e^{i\theta(s)} = \left(\frac{\sum_{j\in\mathrm{N}_n} b_j e^{ijs}}{2\prod_{j\in\mathrm{N}_n}(1-\lambda_j e^{is})} + c\right)\prod_{j\in\mathrm{N}_n}\frac{e^{is}-\lambda_j}{1-\lambda_j e^{is}}$$
$$+\frac{\sum_{j\in\mathrm{N}_n}\overline{b_j}e^{i(n-j)s}}{2\prod_{j\in\mathrm{N}_n}(1-\lambda_j e^{is})}, \quad s\in[-\pi,\pi].$$

It is clear that the above equation implies that (2.29) holds for some $[\alpha_j : j\in\mathbb{Z}_+]\in l^2(\mathbb{Z}_+)$. Conversely, if (2.29) holds for some $[\alpha_j : j\in\mathbb{Z}_+]\in l^2(\mathbb{Z}_+)$ then there exists $[\alpha'_j : j\in\mathbb{Z}_+]\in l^2(\mathbb{Z}_+)$ such that

$$\left(\sum_{j\in\mathrm{N}} c_{-j}(\rho)e^{-ijs}\right)\prod_{j\in\mathrm{N}_n}\frac{e^{is}-\lambda_j}{1-\lambda_j e^{is}} = \sum_{j\in\mathrm{Z}_+}\alpha'_j e^{ijs}, \quad s\in[-\pi,\pi]. \tag{2.31}$$

The same reasoning as that used in the proof of Theorem 2.1 then yields by (2.31) that there exists $b'_j\in\mathbb{C}$, $j\in\mathbb{N}_n$ such that

$$\sum_{j\in\mathrm{N}} c_{-j}(\rho)e^{-ijs} = \frac{\sum_{j\in\mathrm{Z}_n} b'_j e^{ijs}}{\prod_{j\in\mathrm{N}_n}(e^{is}-\lambda_j)}, \quad s\in[-\pi,\pi]. \tag{2.32}$$

Noting that (2.32) implies that ρ has the form (2.30) for some $b_j\in\mathbb{C}$, $j\in\mathbb{N}_n$ and $c\in\mathbb{R}$ completes the proof. □

3 Orthonormal bases for $L^2_{\mathrm{r}}(\ )$

We present two constructions of orthonormal bases for $L^2_{\mathrm{r}}(\ )$ with the basis functions in $\mathcal{M}$. Our rst result below shows that this task can be reduced to a construction of orthonormal bases for the Hard space $\mathbf{H}^2(\ _+)$.

Theorem 3.1 *Functions* $M_j \in \mathbf{H}^2(\ _+)$, $j \in \ _+$, *with nontangential boundary limits*

$$M_j(t) = \rho_j(t)e^{i\theta_j(t)}, \quad t \in \ , \quad j \in \ _+ \tag{3.1}$$

form an orthonormal basis for $\mathbf{H}^2(\ _+)$ *if and only if* $\sqrt{2}\rho_j \cos\theta_j$, $\sqrt{2}\rho_j \sin\theta_j$, $j \in \ _+$, *satisfy*

$$H(\rho_j(\cdot)\cos\theta_j(\cdot))(t) = \rho_j(t)\sin\theta_j(t), \; t \in \ , \quad j \in \ _+ \tag{3.2}$$

and constitute an orthonormal basis for $L^2_r(\ )$.

Proof Suppose that M_j, $j \in \ _+$, with the amplitude-phase modulation (3.1), form an orthonormal basis for $\mathbf{H}^2(\ _+)$. Let f be an arbitrar function in $L^2_{\mathrm{r}}(\ )$. B Lemma 2.2, $f + iHf$ is the nontangential boundar limit of some function in $\mathbf{H}^2(\ _+)$. There hence e ists $[\alpha_j : j \in \ _+] \in \ ^2(\ _+)$ such that

$$f(t) + i(Hf)(t) = \sum_{j \in \mathrm{Z}_+} \alpha_j M_j(t), \quad t \in \ ,$$

where the equalit holds in $L^2(\ )$. The above equation implies that the linear span of $\mathcal{A} := \{\sqrt{2}\rho_j \cos\theta_j, \sqrt{2}\rho_j \sin\theta_j : j \in \ _+\}$ is dense in $L^2_{\mathrm{r}}(\ )$. It remains to prove the orthonormalit of $\mathcal{A}$. To this end, we note that since $\rho_j e^{i\theta_j}$ is the nontangential boundar limit of $M_j \in \mathbf{H}^2(\ _+)$ there holds for each $j \in \ _+$ that

$$H(\rho_j(\cdot)\cos\theta_j(\cdot))(t) = \rho_j(t)\sin\theta_j(t), \quad H(\rho_j(\cdot)\sin\theta_j(\cdot))(t) = -\rho_j(t)\cos\theta_j(t), \quad t \in \ .$$

B (1.1), we obtain that

$$\mathcal{F}(\rho_j \sin\theta_j)(\xi) = -i\mathrm{sgn}(\xi)\mathcal{F}(\rho_j \cos\theta_j)(\xi), \quad \xi \in \ . \tag{3.3}$$

Consequentl , there holds for each $j, k \in \ _+$

$$\begin{aligned}\langle M_j, M_k \rangle_{\mathbf{H}^2(\mathrm{C}_+)} &= \int_{\mathrm{R}} \mathcal{F}(\rho_j e^{i\theta_j})(\xi)\overline{\mathcal{F}(\rho_k e^{i\theta_k})(\xi)}d\xi \\ &= 4\int_0^\infty \mathcal{F}(\rho_j \cos\theta_j)(\xi)\overline{\mathcal{F}(\rho_k \cos\theta_k)(\xi)}d\xi.\end{aligned}$$

B the assumption that M_j, $j \in \ _+$, form an orthonormal basis for $\mathbf{H}^2(\ _+)$, we get

$$\int_0^\infty \mathcal{F}(\rho_j \cos\theta_j)(\xi)\overline{\mathcal{F}(\rho_k \cos\theta_k)(\xi)}d\xi = \frac{1}{4}\delta_{j,k}, \tag{3.4}$$

where $\delta_{j,k}$ denotes the Kronecker delta. Finally, we calculate by the basic properties of the Fourier transform and (3.3) that

$$\langle \rho_j \cos\theta_j, \rho_k \cos\theta_k \rangle_{L^2(\mathbb{R})} = \langle \rho_j \sin\theta_j, \rho_k \sin\theta_k \rangle_{L^2(\mathbb{R})}$$
$$= 2\,\mathrm{Re}\left(\int_0^\infty F(\rho_j \cos\theta_j)(\xi)\overline{F(\rho_k \cos\theta_k)(\xi)}d\xi \right)$$

and

$$\langle \rho_j \cos\theta_j, \rho_k \sin\theta_k \rangle_{L^2(\mathbb{R})} = -2\,\mathrm{Im}\left(\int_0^\infty F(\rho_j \cos\theta_j)(\xi)\overline{F(\rho_k \cos\theta_k)(\xi)}d\xi \right).$$

The above two equations together with (3.4) prove the orthonormality of A in $L^2_{\mathrm{r}}(\mathbb{R})$.

Conversely, assume that $\sqrt{2}\rho_j \cos\theta_j$, $\sqrt{2}\rho_j \sin\theta_j$, $j \in \mathbb{Z}_+$, satisfy (3.2) for all $j \in \mathbb{Z}_+$ and constitute an orthonormal basis for $L^2_r(\mathbb{R})$. It follows by Lemma 2.2 that $M_j \in \mathbf{H}^2(\mathbb{C}_+)$, $j \in \mathbb{Z}_+$. It is clear that there holds for each $j, k \in \mathbb{Z}_+$

$$\langle M_j, M_k \rangle_{\mathbf{H}^2(\mathbb{C}_+)} = \langle \rho_j \cos\theta_j, \rho_k \cos\theta_k \rangle_{L^2(\mathbb{R})} + \langle \rho_j \sin\theta_j, \rho_k \sin\theta_k \rangle_{L^2(\mathbb{R})} = \delta_{j,k}$$

which confirms the orthonormality of $\{M_j : j \in \mathbb{Z}_+\}$ in $\mathbf{H}^2(\mathbb{C}_+)$. It suffices to prove that the linear span of $\{M_j : j \in \mathbb{Z}_+\}$ is dense in $\mathbf{H}^2(\mathbb{C}_+)$. Set $M \in \mathbf{H}^2(\mathbb{C}_+)$. Suppose that its nontangential boundary limit has the form

$$M(t) = \rho(t)e^{i\theta(t)}, \quad t \in \mathbb{R},$$

where $\rho \geq 0$ and θ is real. We can find $[\alpha_j : j \in \mathbb{Z}_+], [\beta_j : j \in \mathbb{Z}_+] \in \ell^2(\mathbb{Z}_+)$ such that there holds the equality in $L^2(\mathbb{R})$

$$\rho(t)\cos\theta(t) = \sum_{j\in\mathbb{Z}_+} \alpha_j \rho_j(t)\cos\theta_j(t) + \sum_{j\in\mathbb{Z}_+} \beta_j \rho_j(t)\sin\theta_j(t), \; t \in \mathbb{R}. \tag{3.5}$$

Applying the Hilbert transform to both sides of the above equation yields that

$$\rho(t)\sin\theta(t) = \sum_{j\in\mathbb{Z}_+} \alpha_j \rho_j(t)\sin\theta_j(t) - \sum_{j\in\mathbb{Z}_+} \beta_j \rho_j(t)\cos\theta_j(t), \; t \in \mathbb{R}, \tag{3.6}$$

where the equation also holds in $L^2(\mathbb{R})$. Combining (3.5) and (3.6) follows that there holds in $L^2(\mathbb{R})$

$$M(t) = \sum_{j\in\mathbb{Z}_+} (\alpha_j - i\beta_j)\rho_j(t)e^{i\theta_j(t)}.$$

This equation implies that the linear span of $\{M_j : j \in \mathbb{Z}_+\}$ is dense in $\mathbf{H}^2(\mathbb{C}_+)$ and proves the theorem. □

Motivated by Theorem 3.1, we next consider the construction of orthonormal bases $\{M_j : j \in \mathbb{Z}_+\}$ for $\mathbf{H}^2(\mathbb{C}_+)$. Our first construction makes use of the *outer functions* in $\mathbf{H}^2(\mathbb{C}_+)$. Those are functions $h \in \mathbf{H}(\mathbb{C}_+)$ of the form

$$h(z) = \exp\{u(z) + iv(z)\}, \quad z \in \mathbb{C}_+,$$

where

$$u(z) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}(\log\phi)(t)\mathrm{d}t, \quad z = x+iy \in \mathbb{C}_+$$

with ϕ being a nonnegative function such that

$$\int_R \frac{|(\log\phi)(t)|\mathrm{d}t}{1+t^2} < \infty,$$

and where v is a harmonic conjugate function of u, [9]. A function $f \in \mathbf{H}^2(\mathbb{C}_+)$ is an outer function if and only if the linear span

$$\operatorname{span}\left\{f(\cdot)e^{iy\cdot} : y \geq 0\right\}$$

is dense in $\mathbf{H}^2(\mathbb{C}_+)$, [12]. There is another characterization of outer functions in $\mathbf{H}^2(\mathbb{C}_+)$, [9]. It states that $f \in \mathbf{H}^2(\mathbb{C}_+)$ is an outer function if and only if there holds

$$\log|f(i)| = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|f(t)|}{1+t^2}\mathrm{d}t. \tag{3.7}$$

For an $f \in \mathbf{H}^\infty(\mathbb{C}_+)$ we denote by $\mathbf{H}^2_f(\mathbb{C}_+)$ the Hilbert space completed upon the linear space of functions in $\mathbf{H}^2(\mathbb{C}_+)$ under the inner product

$$\langle g, h\rangle_{\mathbf{H}^2_f(\mathbb{C}_+)} := \int_{\mathbb{R}} g(t)\overline{h(t)}|f(t)|^2\mathrm{d}t, \quad g, h \in \mathbf{H}^2(\mathbb{C}_+). \tag{3.8}$$

Theorem 3.2 *Suppose that $f_1, f_2 \in \mathbf{H}^2(\mathbb{C}_+)$ satisfy that $f_1/f_2 \in \mathbf{H}^\infty(\mathbb{C}_+)$ and f_1 is an outer function. If $e_j \in \mathbf{H}^2(\mathbb{C}_+)$, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_{f_1/f_2}(\mathbb{C}_+)$, then $\frac{f_1}{f_2}e_j$, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2(\mathbb{C}_+)$.*

Proof Suppose that all the assumptions are satisfied. We first see that $\frac{f_1}{f_2}e \in \mathbf{H}^2(\mathbb{C}_+)$ whenever $e \in \mathbf{H}^2(\mathbb{C}_+)$. This observation together with the definition (3.8) ensures immediately that $\frac{f_1}{f_2}e_j$, $j \in \mathbb{Z}_+$, form an orthonormal sequence in $\mathbf{H}^2(\mathbb{C}_+)$. It remains to show that their linear span is dense in $\mathbf{H}^2(\mathbb{C}_+)$. Set $y \in [0, \infty)$. By the assumption that e_j, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_{f_1/f_2}(\mathbb{C}_+)$, $\frac{f_1}{f_2}(f_2e^{iy\cdot})$ can be approximated arbitrarily close by the functions in $\operatorname{span}\{\frac{f_1}{f_2}e_j : j \in \mathbb{Z}_+\}$ in the Hilbert space $\mathbf{H}^2(\mathbb{C}_+)$. This fact reveals that $f_1e^{iy\cdot}$ is contained in the closure of $\operatorname{span}\{\frac{f_1}{f_2}e_j : j \in \mathbb{Z}_+\}$ in $\mathbf{H}^2(\mathbb{C}_+)$. Noting that f_1 is an outer function completes the proof. □

To give a concrete example for Theorem 3.2, we fix $a \in \mathbb{C}_+$ and introduce

$$f_1(z) := \frac{\sqrt{1-|K(a)|^2}}{1+\overline{K(a)}}\frac{1}{z-\bar{a}}, \quad f_2(z) := \frac{1}{1-iz}, \quad z \in \mathbb{C}_+. \tag{3.9}$$

Clearly, f_1, f_2 belong to $\mathbf{H}^2(\mathbb{C}_+)$. One may use the following Jensen formula (see, [22], page 59) to verify if an analytic function is an outer function.

[redacted]mma 3.3 *Set* $0 < r < R$*. If* g *is analytic in* $\{z \in \mathbb{C} : |z| < R\}$, $g(0) \neq 0$ *and* z_j, j [redacted], *are the zeros of* g *in* $\{z \in \mathbb{C} : |z| \leq r\}$ *then there holds*

[redacted]

If a [redacted]
ther [redacted]

Pro [redacted]

Pro [redacted]
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re [redacted]
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M [redacted]

Set $b := K(a)$. We get b (3.11) that

$$\langle e_j, e_k\rangle_{\mathbf{H}^2_{f_1/f_2}(\mathrm{C}_+)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{it}-b}{1-\bar{b}\,e^{it}}\right)^{j-k}\frac{1-|b|^2}{|1-\bar{b}\,e^{it}|^2}\mathrm{d}t. \tag{3.12}$$

De ne

$$m(z) := \frac{z-b}{1-\bar{b}\,z}, \quad z \in \ .$$

Appl ing the change of variables $e^{is} = m(e^{it})$ to the integral in the right hand side of (3.12) ields that

$$\langle e_j, e_k\rangle_{\mathbf{H}^2_{f_1/f_2}(\mathrm{C}_+)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j-k)s}ds = \delta_{j,k}.$$

We conclude that e_j, $j \in {}_+$, are orthonormal in $\mathbf{H}^2_{f_1/f_2}(\ _+)$. To complete the proof, it suf ces to show that their linear span is dense in $\mathbf{H}^2_{f_1/f_2}(\ _+)$. Suppose that $f \in \mathbf{H}^2_{f_1/f_2}(\ _+)$ is orthogonal to e_j in $\mathbf{H}^2_{f_1/f_2}(\ _+)$ for all $j \in {}_+$. Then similar arguments as those engaged to obtain (3.11) are able to prove that

$$g(z) := \left[T^{-1}\left(f\frac{f_1}{f_2}\right)\right]\circ m^{-1}(z), \ z \in$$

is orthogonal to z^j in $\mathbf{H}^2(\ )$ for all $j \in {}_+$. Therefore, f is a trivial function. □

B Theorem 3.2, (3.10) and Propositions 3.4, 3.5,

$$\frac{1}{\sqrt{\pi}}\frac{\sqrt{1-|K(a)|^2}}{1+\overline{K(a)}}\left(\frac{\bar{a}-i}{a+i}\right)^j\left(\frac{z-a}{z-\bar{a}}\right)^j\frac{1}{z-\bar{a}}, \quad j \in {}_+ \tag{3.13}$$

form an orthonormal basis for $\mathbf{H}^2(\ _+)$. We shall transform it into one for $L^2_{\mathrm{r}}(\ )$ b Theorem 3.1. Set $a_r := \mathrm{Re}\,(a)$, $a_i := \mathrm{Im}\,(a)$, $b := K(a)$ and $\mu \in$ such that

$$\nu := \frac{\mu}{\sqrt{\pi}}\frac{\sqrt{1-|K(a)|^2}}{1+\overline{K(a)}} > 0.$$

We also denote for each $\lambda \in$ b ζ_λ the real function on $[-\pi, \pi]$ de ned b

$$\frac{e^{is}-\lambda}{1-\bar{\lambda}e^{is}} = e^{i\zeta_\lambda(s)}, \quad s \in [-\pi, \pi].$$

It has a positive derivative as shown e plicitl below

$$\zeta_\lambda'(s) = \frac{1-|\lambda|^2}{1-2\,\mathrm{Re}\,(\lambda e^{-is})+|\lambda|^2}, \quad s \in (-\pi, \pi).$$

Multipl ing each of the functions (3.13) b the constant μi and calculating the nontangential boundar limits of the resulting new functions ields that for amplitudes and phases given b

$$\gamma_j(t) := \frac{\nu}{\sqrt{(t-a_r)^2+a_i^2}}, \ \theta_j(t) := j\zeta_b(2\arctan t) + \arctan\frac{t-a_r}{a_i}, \quad t \in \ , \quad j \in {}_+,$$

functions

$$(\gamma_j \cos\theta_j)(t) = \frac{\nu a_i}{(t-a_r)^2 + a_i^2}\cos(j\zeta_b(2\arctan t)) + \frac{\nu(a_r - t)}{(t-a_r)^2 + a_i^2}\sin(j\zeta_b(2\arctan t)),\ t \in \quad,\ j \in \quad_+$$

and

$$(\gamma_j \sin\theta_j)(t) = \frac{\nu(t - a_r)}{(t-a_r)^2 + a_i^2}\cos(j\zeta_b(2\arctan t)) + \frac{\nu a_i}{(t-a_r)^2 + a_i^2}\sin(j\zeta_b(2\arctan t)),\ t \in \quad,\ j \in \quad_+$$

form an orthonormal basis for $L^2_{\mathrm{r}}(\quad)$ that satis es (3.2) and $\theta_j' > 0$, $j \in \quad_+$.

4 A second construction

Our second construction is stimulated b the following simple observation.

Lemma 4.1 *If g is a function in* $\mathbf{H}^\infty(\quad_+)$ *then*

$$\frac{1}{\pi}\int_{\mathsf{R}} \frac{g(t)}{1+t^2}\frac{i-t}{i+t}dt = 0. \tag{4.1}$$

Proof B the change of variables (2.4), we see that

$$\frac{1}{\pi}\int_{\mathsf{R}} \frac{g(t)}{1+t^2}\frac{i-t}{i+t}\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{is}(g\circ K^{-1})(e^{is})ds = \frac{1}{2\pi i}\int_{\partial\mathsf{U}}(g\circ K^{-1})(z)dz. \tag{4.2}$$

B the Cauch integral formula (2.10), we have for each $r \in (0, 1)$ that

$$\frac{1}{2\pi i}\int_{\partial\mathsf{U}}(g\circ K^{-1})(rz)dz = 0. \tag{4.3}$$

Since $g\circ K^{-1} \in \mathbf{H}^\infty(\quad)$, $(g\circ K^{-1})(r\cdot)$ converges in $L^1(\partial\quad)$ to $g\circ K^{-1}$as r goes to 1 (see, [18], page 340). This fact together with (4.3) proves that the last integral in (4.2) vanishes and hence completes the proof. □

The *finite Blaschke product* associated with a nite number of points $z_j \in \quad_+$, $j \in \quad_n$, is the anal tic function f on $\quad_+$ de ned as

$$f(z) := \prod_{j\in\mathsf{N}_n}\frac{z - z_j}{z - \overline{z_j}},\quad z \in \quad_+.$$

Suppose we have a sequence of functions $f_n \in \mathbf{H}^\infty(\ _+)$, $n \in .\ $, with the properties that $f_n(i) = 0$ and

$$\overline{f_n(t)} = \left(\frac{h_n}{g_n}\right)(t), \quad t \in \quad , \quad n \in .\ , \tag{4.4}$$

where h_n, g_n are anal tic functions on $_+$ with g_n having at least one but a nite number of eros in $_+$. Let b_n be the nite Blaschke product associated with the eros of g_n in $_+$, $n \in .\ $. With such a sequence of anal tic functions, we de ne

$$\beta_0(z) := \frac{1}{\sqrt{\pi}}\frac{1}{1-iz}, \quad \beta_n(z) := \frac{1}{\sqrt{\pi}}\frac{1}{1-iz} f_n(z) \prod_{j \in \mathsf{N}_{n-1}} b_j(z), \quad z \in \ _+, \ n \in .\ . \tag{4.5}$$

Here we denote $._0 := \emptyset$.

Theorem 4.2 *The functions β_n, $n \in \ _+$, constructed by (4.5) are orthogonal in $\mathbf{H}^2(\ _+)$.*

Proof We observe that $\beta_n \in \mathbf{H}^2(\ _+)$, $n \in .\ $, because the are products of a function in $\mathbf{H}^2(\ _+)$ and a bounded anal tic function. For $n \in .\ $, we have that

$$\langle \beta_n, \beta_0 \rangle_{\mathbf{H}^2(\mathsf{C}_+)} = \frac{1}{\pi}\int_{\mathsf{R}} \frac{1}{1+t^2} f_n(t) \prod_{j \in \mathsf{N}_{n-1}} b_j(t)\mathrm{d}t,$$

which, b Lemma 4.1, equals to ero since

$$f_n(z) = (i+z)\frac{f_n(z)}{i-z}\frac{i-z}{i+z}, \quad z \in \ _+$$

with $(i+z)\dfrac{f_n}{i-z}$ being bounded and anal tic on $_+$. It is also calculated for $n > m \geq 1$ b (4.4) that

$$\begin{aligned}
\langle \beta_n, \beta_m \rangle_{\mathbf{H}^2(\mathsf{C}_+)} &= \int_{\mathsf{R}} \beta_n(t)\overline{\beta_m(t)}\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\mathsf{R}} \frac{1}{1+t^2} f_n(t)\left(\frac{h_m}{g_m}\right)(t)\left(\prod_{j=m}^{n-1} b_j(t)\right)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\mathsf{R}} \frac{1}{1+t^2} f_n(t)\left(h_m\frac{b_m}{g_m}\right)(t)\left(\prod_{j=m+1}^{n-1} b_j(t)\right)\mathrm{d}t.
\end{aligned}$$

The integral above is also equal to ero since the function

$$f_n h_m \frac{b_m}{g_m}\prod_{j=m+1}^{n-1} b_j$$

is bounded and anal tic on $_+$ with a ero at $z = i$. □

Let us see a simplest e ample of the above construction. In this e ample, h_n in (4.4) is onl to meet the requirement that $f_n(i) = 0$ and g_n has a single ero in $\,_+$. In other words, we set

$$h_n(z) := z + i, \quad g_n(z) := z - d_n, \quad z \in \,_+, \quad n \in . \, ,$$

where $d_n \in \,_+, n \in .$. As a consequence, we have for $z \in \,_+$ that

$$b_n(z) := \frac{z - d_n}{z - \overline{d_n}}$$

and

$$\beta_0(z) := \frac{1}{\sqrt{\pi}} \frac{1}{1 - iz}, \quad \beta_n(z) := \frac{1}{\sqrt{\pi}} \frac{1}{1 - iz} \frac{z - i}{z - \overline{d_n}} \prod_{j \in \mathrm{N}_{n-1}} \frac{z - d_j}{z - \overline{d_j}}, \quad n \in . \, . \tag{4.6}$$

It can be veri ed directl that the phases of the above functions also possess nonnegative derivatives.

Theorem 4.3 *Let β_j, $j \in \;_+$, be constructed as in* (4.6) *where $d_n \in \,_+$, $n \in .$, are pairwise distinct. Then* span $\{\beta_j : j \in \;_+\}$ *is dense in $\mathbf{H}^2(\,_+)$ if and only if*

$$\sum_{n \in \mathrm{N}} (1 - |K(d_n)|) = \infty. \tag{4.7}$$

Proof Let $\phi \in \mathbf{H}^2(\,_+)$. Using the isomorphism (2.5), one can show b induction that ϕ is orthogonal to β_j for all $j \in \;_+$ if and onl if

$$\phi(i) = 0, \quad \phi(d_n) = 0, \quad n \in . \, .$$

It hence suf ces to point out the fact that there does not e ist a nontrivial function in $\mathbf{H}^2(\,_+)$ that vanishes on $\{d_n : n \in . \}$ if and onl if (4.7) holds, [9]. □

We now use Theorems 4.2 and 4.3 to derive orthonormal bases for $L^2_{\mathrm{r}}(\;)$. Choose pairwise distinct $d_n \in \,_+$, $n \in .$, that satisf (4.7). This can be done b , for e ample, requiring that $|K(d_n)| = 1 - n^{-1}$, $n \in .$. Set $d_{n,r} := \mathrm{Re}\,(d_n)$, $d_{n,i} := \mathrm{Im}\,(d_n)$ and $b_n := K(d_n)$, $n \in .$. We modif the construction (4.6) slightl to get that

$$\frac{1}{\sqrt{\pi}} \frac{1}{1 - iz}, \quad \sqrt{\frac{d_{n,i}}{\pi}} \frac{i - z}{z + i} \frac{i}{z - \overline{d_n}} \prod_{j \in \mathrm{N}_{n-1}} \left(\frac{z - d_j}{z - \overline{d_j}} \frac{\overline{d_j} - i}{d_j + i} \right), \quad n \in . \, ,$$

constitute an orthonormal bases for $\mathbf{H}^2(\,_+)$. As discussed in the e ample at the end of the last section, the phase of each of the basis functions has a positive derivative. Finall , we calculate the nontangential boundar limits of the basis functions to conclude that the following functions

$$\frac{1}{\sqrt{\pi}}\frac{1}{1+t^2}, \quad \frac{1}{\sqrt{\pi}}\frac{t}{1+t^2}$$

$$\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,i}}{(t-d_{n,r})^2+d_{n,i}^2}\cos(\omega_n(2\arctan t)) \\ +\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,r}-t}{(t-d_{n,r})^2+d_{n,i}^2}\sin(\omega_n(2\arctan t)),\ n\in .$$

and

$$\sqrt{\frac{d_{n,i}}{\pi}}\frac{t-d_{n,r}}{(t-d_{n,r})^2+d_{n,i}^2}\cos(\omega_n(2\arctan t)) \\ +\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,i}}{(t-d_{n,r})^2+d_{n,i}^2}\sin(\omega_n(2\arctan t)),\ n\in .\ ,$$

form an orthonormal basis for $L^2_{\mathrm{r}}(\ )$, where $\omega_n := \zeta_0 + \sum_{j\in \mathrm{N}_{n-1}} \zeta_{b_j}$, $n \in .\ .$

5 Orthonormal bases for $L^2_{\mathrm{r}}[-\pi,\pi]$

In this section, we present parallel results for the construction of orthonormal bases for $L^2_{\mathrm{r}}[-\pi,\pi]$. We omit the proofs since their arguments are similar to those in the last two sections.

Theorem 5.1 *Functions* $1, m_j,\ j\in .\ $, *with nontangential boundary limits*

$$m_j(e^{it}) = \rho_j(t)e^{i\theta_j(t)},\quad t\in[-\pi,\pi],\quad j\in .\ ,$$

form an orthonormal basis for $\mathbf{H}^2(\ )$ *if and only if* $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}}\rho_j\cos\theta_j$, $\frac{1}{\sqrt{\pi}}\rho_j\sin\theta_j$, $j\in .\ $, *satisfy for each* $j\in .$

$$\tilde{H}(\rho_j(\cdot)\cos\theta_j(\cdot))(t) = \rho_j(t)\sin\theta_j(t),\quad t\in[-\pi,\pi].$$

and constitute an orthonormal basis for $L^2_r[-\pi,\pi]$.

Our ne t result parallels to Theorem 3.2 in Section 3. We call $h \in \mathbf{H}(\ )$ an *outer function* if it is of the form

$$h(z) = c\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}(\log\psi)(e^{it})\mathrm{d}t\right\},\quad z\in\ ,$$

where $c\in\ $, ψ is a positive Lebesgue measurable function on such that $\log\psi\in L^1(\ )$, [18]. According to [3, 9, 18], a function $f\in\mathbf{H}(\ )$ is an outer function if and onl if the linear span of $\{fp_j : j \in \mathbb{Z}_+\}$ is dense in $\mathbf{H}^2(\mathbb{D})$, where $p_j(z) := z^j$. We denote b $\mathbf{H}^2_f(\mathbb{D})$ for an $f \in \mathbf{H}^\infty(\mathbb{D})$ the Hilbert space completed upon the linear space of functions in $\mathbf{H}^2(\mathbb{D})$ endowed with the following inner product

$$\langle g, h\rangle_{\mathbf{H}^2_f(\mathbb{D})} := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})\overline{h(e^{it})}|f(e^{it})|^2 \mathrm{d}t, \quad g, h \in \mathbf{H}^2_f(\mathbb{D}).$$

Theorem 5.2 *Let $f \in \mathbf{H}^2(\mathbb{D})$ be a bounded outer function. If $e_j \in \mathbf{H}^2(\mathbb{D})$, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_f(\mathbb{D})$ then fe_j, $j \in \mathbb{Z}_+$, constitute an orthonormal basis for $\mathbf{H}^2(\mathbb{D})$.*

We also have a construction similar to the one in Section 4. A *finite Blaschke product* $f \in \mathbf{H}^2(\mathbb{D})$ associated with $\{z_j : j \in \mathbb{N}_n\} \subseteq \mathbb{D}$ is the function

$$f(z) := \prod_{j\in\mathbb{N}_n} \frac{z - z_j}{1 - \overline{z_j}z}, \quad z \in \mathbb{D}.$$

Theorem 5.3 *Suppose that f_n, $n \in \mathbb{N}$, is a sequence of functions in $\mathbf{H}^\infty(\mathbb{D})$ with the properties that for each $n \in \mathbb{N}$, $f_n(0) = 0$ and there exist analytic functions h_n, g_n on $\mathbb{D}$ such that*

$$\overline{f_n(e^{it})} = \left(\frac{h_n}{g_n}\right)(e^{it}), \quad t \in [-\pi, \pi]$$

and g_n has at least one but a finite number of zeros in $\mathbb{D}$. Then

$$m_n(z) := f_n(z) \prod_{j\in\mathbb{N}_{n-1}} b_j(z), \quad z \in \mathbb{D}, \; n \in \mathbb{N},$$

are orthogonal in $\mathbf{H}^2(\mathbb{D})$, where b_n is the finite Blaschke product associated with the zeros of g_n.

Particular e amples of bases for $L^2_{\mathrm{r}}[-\pi, \pi]$ that t into the general constructions described b the last two theorems can be found in ([5] and Wang et al., preprint).

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