HILBERT TRANSFORMS AND THE CAUCHY INTEGRAL IN EUCLIDEAN SPACE

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Abstract. We generalize the notions of harmonic conjugate functions and Hilbert transforms to higher dimensional euclidean spaces, in the setting of differential forms and the Hodge-Dirac system. These harmonic conjugates are in general far from being unique, but under suitable boundary conditions we prove existence and uniqueness of conjugates. The proof also yields invertibility results for a new class of generalized double layer potential operators on Lipschitz surfaces and boundedness of related Hilbert transforms.

1. Introduction

... D ... $\mathbf{R}^2$ ... $\mathbf{C}$ ... $u(z)$ ... D *harmonic conjugate* ... $u(z)$ ... $v(z)$ ... D ... $f(z) = u(z) + iv(z)$ ... $v(z)$ ... v ... D ... $u \mapsto v$ ... *Hilbert transform*

$$H_D : u|_{\partial D} \mapsto v|_{\partial D}$$

... $D \subset \mathbf{R}^2$ ... D ... D ... $\mathbf{C} \setminus \overline{D}$ ... $h : \partial D \to \mathbf{C}$ ...

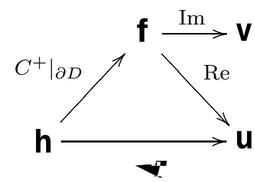
$$F(z) = C^{\pm}h(z) = \pm\frac{1}{2\pi i}\int_{\partial D}\frac{h(w)}{w-z}dw, \qquad z \in D^{\pm},$$

... D^+ ... D ... D^- ... $\mathbf{C} \setminus \overline{D}$ ... $f^{\pm}(\zeta)$ ... $\lim_{\to \zeta, \in D^{\pm}} F(z)$ ... f^+ ... f^- ... h ... ∂D ... f^+ ... f^- ... f^- ... f ... f^+ ... u ... iv ... f ... u ... v ... ∂D ... h ...

h

h

$f = C^+h|_{\partial D} = \frac{1}{2}h + \frac{1}{2\pi i}\int_{\partial D} h(w)(w-z)^{-1}dw$



double layer potential operator

$$Th(z) := 2\,\mathrm{Re}\,C^+h|_{\partial D}(z) - h(z) = \frac{1}{\pi}\int_{\partial D}\mathrm{Im}\left(\frac{dw}{w-z}\right)h(w), \quad z \in \partial D,$$

$$v = H_D u = \mathrm{Im}\,C^+(T^{-1}u)|_{\partial D}.$$

D $(w-z)$ D T^{-1}

d δ

$$dF(x) = \nabla \wedge F(x) = \sum_{j=1}^{n} e_j \wedge \partial_j F(x),$$

$$\delta F(x) = \nabla \lrcorner F(x) = \sum_{j=1}^{n} e_j \lrcorner \partial_j F(x),$$

$\langle x, u \wedge y \rangle = \langle u \lrcorner x, y \rangle$ $u, x, y \in \wedge \mathbf{R}^n$. F $\wedge \mathbf{R}^n$ k $F = \sum F$ $F : D \to \wedge \mathbf{R}^n$

$$\cdots \rightleftarrows \wedge^{k-2}\mathbf{R}^n \rightleftarrows \wedge^{k-1}\mathbf{R}^n \rightleftarrows \wedge^{k}\mathbf{R}^n \rightleftarrows \wedge^{k+1}\mathbf{R}^n \rightleftarrows \wedge^{k+2}\mathbf{R}^n \rightleftarrows \cdots$$

$dF_{k-1} = -\delta F_{k+1}$, $\le k \le n$, F *monogenic* ... *harmonic* ... F ... $d\delta^2 F$ ... δd ... $d\delta F$ ... F ... $d^2 = \delta^2 = 0$. $\le k \le n$ k $U : D \to \wedge^k \mathbf{R}^n$, $V_1 : D \to \wedge^{k-2}\mathbf{R}^n$, $V_2 : D \to \wedge^{k+2}\mathbf{R}^n$ ... *harmonic conjugates* ... U ... d ... δ ... V_1 ... U ... V_2 ... $dU = -\delta V_2$, $\delta U = -dV_1$ ... $dV_2 = \delta V_1 = 0$.

Example 1.1. n ... k ... f ... $u + iv$ ... $\mathbf{R} \approx \wedge^0\mathbf{R}^2$ ... $i\mathbf{R} \approx \wedge^2\mathbf{R}^2$ ... $\{e_1, e_2\}$ ... $\mathbf{R}^2$ ... $i \approx e_1 \wedge e_2$ ...

$$(d + \delta)(u + iv) = (e_1 \wedge \partial_1 u + e_2 \wedge \partial_2 u) + (e_1 \lrcorner i\partial_1 v + e_2 \lrcorner i\partial_2 v)$$

$$= (\partial_1 u - \partial_2 v) e_1 + (\partial_2 u + \partial_1 v) e_2 \ldots$$

v ... u ... $V_2 = v e_1 \wedge e_2 : D \to \wedge^2\mathbf{R}^2$ ... u ... V_1 ...

V_1 V_2 ... U *two-sided harmonic* ... δdU ... $d\delta U$ ... V_1 V_2 ... *two-sided monogenic fields* ... $V_1 - V_1'$ ... $V_2 - V_2'$ ... $d(V - V')$ ... $\delta(V - V')$ ... i ... ; ... $\le k \le n -$ ... k ...

V_1 ... V_2 ...

$$U \mapsto V_1, V_2.$$

Cauchy type … U, V_1 … V_2 … ∂D … $U|_D$ … $V_1|_D, V_2|_D$ … D

k … k … n … $\mathbf{R}^n$ … L_p … $< p < \infty$ … $< p < \infty$

k … $\leq k \leq n-$

L_2

Hodge type

B … $D \to \mathbf{R}^3$ … d … δ

Hilbert transforms for scalar functions

D … d … δ … D^2

$$DF(x) = \triangle F(x) = \sum_{j=1}^{n} e_j \triangle \partial_j F(x).$$

$\triangle$ … $\mathbf{R}^n$ … $\mathbf{R}^n$ … $\mathbf{R}$

$v \triangle w$ … $v \lrcorner w$ … $v \wedge w$, … $w \triangle v$ … $w \llcorner v$ … $w \wedge v$

[illegible] $\mathbf{v}$ [illegible] $\mathbf{R}^n$ [illegible] $\mathbf{R}^n$ [illegible] $\mathbf{w}$ [illegible] $\mathbf{R}^2$ [illegible] $\mathbf{R}^n$ [illegible]

[illegible] $\mathbf{w}$, $\mathbf{w}, \mathbf{x}, \mathbf{y}$ [illegible] $\mathbf{R}^n$.

[illegible] $\mathbf{w}_1\mathbf{w}_2$ [illegible] $\mathbf{v}$ [illegible] $\mathbf{R}^n$ [illegible] $\mathbf{R}^n$ $\mathbf{w}$ [illegible] $\mathbf{R}^n$ [illegible]

$\mathbf{vw}$ [illegible] n.

[illegible] $\{\mathbf{e}\}_{=1}^n$ [illegible] $\mathbf{R}^n$ [illegible] $\mathbf{k}$ [illegible] $\mathbf{R}^n$ $\{\mathbf{e}_s\}_{|s|}$ [illegible] $\{\mathbf{e}_s\}_{s\subset}$ [illegible]

$D^\pm \subset \mathbf{R}^n$ … ∂D^+ … ∂D^- … D^+ … D^- … $\mathbf{R}^n \setminus \overline{D^+}$ … D … D^+ … D^-

- *graph domain* $D^+ \ \{x \ x_n > \phi(x_1, \ldots, x_{n-1})\}$ … $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ … D^-
- *interior domain* D^+ … *exterior domain* D^- … D^+

$\nu(y)$ … D^- … $\gamma(y) \subset D$ $y \in$ … $D^\pm$ … c_1

$$\gamma(y) = \gamma(y, D^\pm) := \{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \ ; \ \pm(x_n - y_n) > c_1|x' - y'|\},$$

$y = (y', y_n) \in$ … $y \in$ … y … $\gamma(y, D^\pm)$ … $(x, y) < c_2$ … c_2 … $L_p(\Sigma)$ … $\mathbf{R}^n$ … F … $D^\pm$ … *non-tangential maximal function*

$$N_*(F)(y) := \ _{x\in(y,D^\pm)} |F(x)|, \quad y \in .$$

$L_p(\Sigma)$ … $\mathbf{R}^n$

Theorem 2.2. *Let* $D^\pm$ *be Lipschitz graph, interior or exterior domains, and fix* $< p < \infty$. *Let* $h \in L_p(\Sigma)$ *and define the monogenic field*

$$C^\pm h(x) := \pm \int_\Sigma E(y - x)\nu(y)h(y)\, d\sigma(y), \quad x \in D^\pm.$$

Then $\|N_*(C^+h)\|_p$ $\|N_*(C^-h)\|_p \le C\|h\|_p$ *for some* $C < \infty$ *depending only on* p *and the Lipschitz constants for the graphs describing* .

The principal value Cauchy integral

$$Eh(x) := \ p.v.\int_\Sigma E(y - x)\nu(y)h(y)\, d\sigma(y), \quad x \in ,$$

exists a.e. and defines a bounded operator $E : L_p(\Sigma) \to L_p(\Sigma)$ *such that* $E^2 = I$. *The boundary traces* $f^+(z) := \ _{x\to \ , x\in(D^+)} C^+h(x)$ *and* $f^-(z) := \ _{x\to \ , x\in(D^-)} C^-h(x)$ *exist for a.a.* $z \in$ *and in* L_p, *and*

$$E^+h := f^+ = \tfrac{1}{2}(h + Eh) \quad \text{and} \quad E^-h := f^- = \tfrac{1}{2}(h - Eh)$$

define L_p*-bounded projection operators.*

h L_p k D^+

$$F \quad V_1 \quad U \quad V_2 \quad C^+h \quad D^+ \longrightarrow \; {}^{-2}\mathbf{R}^n \quad \mathbf{R}^n \quad {}^{+2}\mathbf{R}^n$$

D^+

$vh \to {}^{-1}\mathbf{R}^n \quad {}^{+1}\mathbf{R}^n$

$E(y - x)$

$$C^+h(x) = \int_\Sigma E(y-x) \lrcorner (\nu(y) \lrcorner h(y)) \, d\sigma(y)$$

$$\int_\Sigma \Big(E(y-x) \wedge (\nu(y) \lrcorner h(y)) \quad E(y-x) \lrcorner (\nu(y) \wedge h(y)) \Big) d\sigma(y)$$

$$\int_\Sigma E(y-x) \wedge \nu(y) \wedge h(y) \, d\sigma(y) \quad V_1 \quad U \quad V_2.$$

Definition 2.3. $U: D^\pm \to \mathbf{R}^n$ δdU $d\delta U$ V_1, V_2 *Cauchy type harmonic conjugates* U $h \to \mathbf{R}^n$ U $C^\pm h$ V_1 $C^\pm h$ $_{-2}$ V_2 $C^\pm h$ $_{+2}$ k k h *dipole density* V_1, U, V_2

$$U|_\Sigma \mapsto V_1|_\Sigma, V_2|_\Sigma,$$

Hilbert transform D

k , h

Theorem 2.4. *Let* $D \subset \mathbf{R}^n$ *be a Lipschitz graph, interior or exterior domain and assume that* $\leq p < \infty$.

(i) *Let* $U: D \to \mathbf{R}$ ${}^0\mathbf{R}^n$ *be a harmonic function such that* $N_*(U) \in L_p$. *If* D *is an exterior domain, also assume that* $_{x\to\infty} U$ *and has trace* $u = U|_\Sigma$ *such that* $\int u\psi \, d\sigma(y)$, *where* ψ *is the function from Theorem 2.5. Then there is a unique Cauchy type harmonic conjugate* $V = V_2: D \to {}^2\mathbf{R}^n$ *to* U, *and a dipole density* $h \in L_p$, *such that*

$$\|N_*(V)\|_p \quad \|h\|_p \lesssim \|N_*(U)\|_p.$$

If D *is a graph or an interior domain, then* h *is unique, and if* D *is an exterior domain, then* h *is unique modulo constants.*

(ii) *In the case* $k = n$, *(i) remains true when* $U: D \to {}^0\mathbf{R}^n$ $\mathbf{R}$ *and* $V = V_2:$ $D \to {}^2\mathbf{R}^n$ *are replaced by* $U: D \to {}^n\mathbf{R}^n \approx \mathbf{R}$ *and* $V = V_1: D \to {}^{n-2}\mathbf{R}^n$.

k k n

$\leq k \leq n -$

principal value double layer potential operator

$$Kh(x) = \int_\Sigma \langle E(y-x), n(y) \rangle h(y) \, d\sigma(y) \quad Eh(x)_0,$$

$h \to \mathbf{R}$, $x \in$ K L_p $< p < \infty$

$I \pm K$

[illegible] p [illegible] L_2 [illegible] $< p < \infty$ [illegible] H^1 [illegible]

Theorem 2.5. *Assume that* [illegible] $\leq p < \infty$ *and let* [illegible] *be a Lipschitz graph or the boundary of an interior / exterior domain. Then*

$$I \quad K: L_p[\text{illegible}] \longrightarrow L_p[\text{illegible}] \; h \mapsto h \quad E h_0 \quad E^+ h_0$$

is an isomorphism. This is also true for $I - K$ $h \mapsto E^- h_0$ *in the case of a graph domain. In the case of an exterior domain,* $I - K$ *is a Fredholm operator with null space consisting of constant functions and range consisting of all* $u \in L_p$ [illegible] *such that* $\int_\Sigma u\psi$ [illegible] *for some* $\psi \in L_q$ [illegible]*, where* $/p$ $/q$ [illegible].

[illegible] $I \pm K$ [illegible]

Proof of Theorem 2.4. [illegible] $h \in L_p$ [illegible] U [illegible] $C^\pm h_0$ [illegible] u [illegible] $U|_\Sigma$ [illegible] $\frac{1}{2} I \pm K\, h \in L_p$ [illegible] h [illegible] u [illegible] u [illegible] V [illegible] $C^\pm h_2$ [illegible] C^- [illegible]

[illegible] $C^\pm h$ [illegible] V U [illegible] h [illegible] $\to \wedge^n \mathbf{R}^n$ $V: D \to \wedge^{n-2}\mathbf{R}^n$ [illegible] $U: D \to \wedge^n \mathbf{R}^n$ [illegible] $U \mapsto U e_n$ [illegible] k [illegible] $n-k$ [illegible] $C^\pm h e_n$ [illegible] $U e_n$ $V e_n$ [illegible] $\wedge^0\mathbf{R}^n$ $\wedge^0\mathbf{R}^n$ [illegible] $\wedge^2\mathbf{R}^n$ [illegible] $U \mapsto U e_n$ [illegible] □

[illegible]

Corollary 2.6. *Let* $V: D^+ \to \wedge^2\mathbf{R}^n$ *be the Cauchy type harmonic conjugate to the harmonic function* $U: D^+ \to \wedge^0\mathbf{R}^n$ [illegible] $\mathbf{R}$*, with suitable estimates of non-tangential maximal functions. Then*

$$U \quad V \; C^+ \; I \quad K^{-1} u\,,$$

where C^+ *is the (interior) Cauchy integral,* K *is the double layer potential operator, and* u [illegible] $U|_\Sigma$*. Taking the trace* v [illegible] $V|_\Sigma$ *of the conjugate function, the Cauchy type Hilbert transform of* u *is*

$$u \mapsto v \quad I \quad E \; I \quad K^{-1} u.$$

Replacing C^+*,* $I \quad K$ *and* $I \quad E$ *with* C^-*,* $I - K$ *and* $I - E$*, the corresponding formulae hold for the domain* D^-*.*

Example 2.7. [illegible] D [illegible] $\mathbf{R}^n_+$ [illegible] $\mathbf{R}^{n-1}$ [illegible] K [illegible] $E(y-x)$ [illegible] $\nu(y)$ [illegible] h [illegible] u [illegible]

$$V(x) [illegible] C^+u [illegible] (x) [illegible] e_n \wedge \frac{[illegible]}{\sigma_{n-1}} \int_{\mathbf{R}^{n-1}} \frac{y-x}{|y-x|^n} u(y)\, dy, \qquad x \in \mathbf{R}^n_+,$$

[illegible] e_n [illegible] $\mathbf{R}^{n-1}$ [illegible] $U|_{\mathbf{R}^{n-1}}$ [illegible] $V|_{\mathbf{R}^{n-1}}$ [illegible] u [illegible] $E u$ [illegible]

[illegible] $\mathbf{R}^n_+$ [illegible] e_n [illegible] $\partial\mathbf{R}^n_+$ [illegible] $\mathbf{R}^{n-1}$ [illegible] F [illegible] $\mathbf{R}^n_+$ [illegible]

$$F(x) [illegible] U(x) e_n [illegible] U(x),$$

[illegible] U [illegible] U [illegible] U [illegible] F [illegible] F [illegible] U [illegible] $U e_n$ [illegible] $\mathbf{R}^n_+ \to$ [illegible] $\mathbf{R}^n$ [illegible] $\mathbf{R}^n$ [illegible]

$$D(U [illegible] U e_n) [illegible] D(U e_n [illegible] U e_n) [illegible] D(U e_n [illegible] U e_n) [illegible].$$

[illegible] $\mathbf{R}^n_+$ [illegible] $U e_n$ [illegible] U [illegible] V [illegible] $U e_n - V$ [illegible] $V(x)$ [illegible] $e_n \wedge \int_{\mathbf{R}^n} E(y-x) u(y)\, dy$ [illegible] $e_n \wedge (U e_n - V)|_{\mathbf{R}^{n-1}}$ [illegible] $U e_n - V$ [illegible]

[illegible] $U e_n$ [illegible] C^+u [illegible] U [illegible]

$$U(x) [illegible] \frac{[illegible]}{\sigma_{n-1}} \int_{\mathbf{R}^n_+} \frac{x'-y}{|x-y|^n} u(y)\, dy, \qquad x [illegible] x', x_n [illegible] \in \mathbf{R}^n_+,$$

[illegible] U [illegible] $n-$ [illegible] $U|_{\mathbf{R}^{n-1}}$ [illegible] u [illegible] D [illegible] e_n [illegible]

[illegible] D [illegible]

$$K h(x) [illegible] \frac{}{\pi} [illegible] \int_\Sigma \frac{\langle y-x, y\rangle}{|y-x|^2} h(y)\, d\sigma(y) [illegible] h,$$

[illegible] h [illegible] h [illegible] I [illegible] $K)^{-1}u$ [illegible] u [illegible] u [illegible]

$$V(x) [illegible] C^+ [illegible] I [illegible] K)^{-1} u [illegible] (x) [illegible] C^+ u [illegible] u [illegible] C^+ u [illegible] (x)$$

$$[illegible] \frac{}{\pi} \int_\Sigma \frac{y \wedge x}{|y-x|^2} u(y)\, d\sigma(y) \longrightarrow [illegible] E u [illegible] e^{it} [illegible] \frac{i}{[illegible]\pi} [illegible] \int_0^{2[illegible]} [illegible] t - s [illegible] u(s)\, ds,$$

$x \to e^t y$, e^s i $e_1 e_2$ L_p $< p < \infty$

n

$I - K$ h h h $I - K$

ψ

$\mathbf{R}^n$ K

$$Kh(x) \quad \frac{\quad}{\ ^{n/2}\sigma_{n-1}} \int_{\Sigma} \frac{h(y)}{- \langle x, y \rangle\ ^{n/2-1}}\, d\sigma(y).$$

L_p

$E(y - x)$ y x

K

L_p $< p < \infty$ C^1

3. Dirac boundary value problems

k

$N(T)$, $R(T)$, $D(T)$

T

L_2 L_2 L_2 L_2 $\ldots$ L_2 L_2

$$L_2 \quad E^+L_2 \oplus E^-L_2 \quad N^+L_2 \oplus N^-L_2.$$

L_2 $E^+L_2 \oplus E^-L_2$ $E^\pm L_2$

E^+L_2

D^+ E^-L_2 D^-

$E^\pm L_2$ $R(E^\pm)$

$E^\pm$ L_2

f $F|_\Sigma \in E^\pm L_2$

F $C^\pm f$ $D^\pm \to \mathbf{R}^n$ F f F

L_2 $N^+L_2 \oplus N^-L_2$

N^+L_2 N^-L_2

Definition 3.1. f $\to \mathbf{R}^n$ *tangential* $\nu(x) \lrcorner f(x)$ x *normal* $\nu(x) \wedge f(x)$ x

$N^\pm$

$$N^+g \quad \nu \lrcorner (\nu \wedge g) \qquad N^-g \quad \nu \wedge (\nu \lrcorner g).$$

[illegible] $\|g_+ \; g_-\|^2 \; \|g_+\|^2 \; \|g_-\|^2$ [illegible] $g_\pm \in N^\pm L_2$ [illegible] $\|f_+ \; f_-\| \approx \|f_+\| \; \|f_-\|$ [illegible] $f_\pm \in E^\pm L_2$ [illegible] $E^+L_2 \cap E^-L_2$ [illegible] $N^+L_2 \cap N^-L_2$ [illegible]

Example 3.2. [illegible] F [illegible] $D^+ \to \mathbb{R}^n$ [illegible] d [illegible] F [illegible] D^+ [illegible] g [illegible] f [illegible] $F|_\Sigma \in E^+L_2$ [illegible] $N^- f$ [illegible] g [illegible] f [illegible] g [illegible]

$$N^- : E^+L_2 \longrightarrow N^-L_2$$

[illegible]

[illegible] $N^-|_{E^+L_2}$ [illegible]

Theorem 3.3. *Let* [illegible] *be any strongly Lipschitz surface. Then the restricted projection* $N^- : E^+L_2 \to N^-L_2$ *is a Fredholm operator of index* [illegible]*, i.e. has closed range and finite dimensional kernel and cokernel of equal dimensions. The same is true for all eight restricted projections*

$$N^+ : E^\pm L_2 \longrightarrow N^+L_2, \qquad N^- : E^\pm L_2 \longrightarrow N^-L_2,$$
$$E^+ : N^\pm L_2 \longrightarrow E^+L_2, \qquad E^- : N^\pm L_2 \longrightarrow E^-L_2.$$

If [illegible] *is a Lipschitz graph, then all these maps are isomorphisms.*

[illegible] θ [illegible] $\langle \nu(x), \theta(x)\rangle \geq c > 0$ [illegible] $x \in$ [illegible]

$$|f|^2 \langle \nu, \theta \rangle \; |\widehat{f}|^2 \tfrac{1}{2} (\nu\theta \;\; \theta\nu) \; \langle \widehat{f}\nu, \widehat{f}\theta \rangle \; \langle \nu \lrcorner f \;\; \nu f, \widehat{f}\theta \rangle,$$

[illegible] $f \mapsto \widehat{f}$ [illegible] k [illegible] k [illegible]

$$\int_\Sigma |f|^2 \langle \nu, \theta \rangle \, d\sigma(y) \; \int_\Sigma \langle \nu \lrcorner f, \widehat{f}\theta \rangle \, d\sigma(y) \quad \sum_{j=1}^{n} \iint_{D^+} \langle F, e_j \widehat{F} \partial_j \theta \rangle \, dx,$$

[illegible] f [illegible] $F|_\Sigma \in E^+L_2$ [illegible] $\|f\| \lesssim \|N^- f\| \;\; \|F\|_{L_2(\mathrm{supp})}$ [illegible] θ [illegible] $-e_n$ [illegible] $N^- : E^+L_2 \to N^-L_2$ [illegible] $f \mapsto F$ [illegible] $N^- : E^+L_2 \to N^-L_2$ [illegible]

[illegible]

Theorem 3.4 [illegible]. *Let* X *and* Y *be Banach spaces, and assume that* $T_\lambda : X \to Y$, $\lambda \in [\ldots]$, *is a family of bounded operators depending continuously on* λ. *If* T_λ *are all semi-Fredholm operators, i.e. has closed range and finite dimensional null space, then the index, i.e.* $\dim(Y/R(T_\lambda)) - \dim N(T_\lambda)$, *of all operators* T_λ *are equal.*

[illegible] $L_2(\ldots)$ [illegible] $E^\pm$ [illegible] $N^\pm$ [illegible]

Definition 3.5. [illegible] $L_2(\ldots)$ [illegible] $D(\ldots) \subset L_2(\ldots)$ [illegible]

[illegible] f [illegible] f_+ [illegible] f_- [illegible] $F_+|_\Sigma$ [illegible] $F_-|_\Sigma \in E^+L_2 \oplus E^-L_2$ [illegible] $F^\pm$ [illegible] $D^\pm$

$$f \ldots dF_+|_\Sigma \ldots dF_-|_\Sigma \ldots -\delta F_+|_\Sigma \ldots -\delta F_-|_\Sigma.$$

[illegible] f [illegible] f_1 [illegible] $\nu \wedge f_2 \in N^+L_2 \oplus N^-L_2$ [illegible] $f_1, f_2 \in N^+L_2$ [illegible]

$$f \ldots d_\Sigma f_1 \ldots \nu \wedge \ldots \delta_\Sigma f_2 \ldots,$$

[illegible] d_Σ [illegible] δ_Σ [illegible]

[illegible] $\rho : \mathbf{R}^{n-1} \to \ldots$ [illegible] $y \in \ldots$ [illegible] ρ^* [illegible] N^+L_2 [illegible] $L_2(\mathbf{R}^{n-1}; \wedge \mathbf{R}^{n-1})$ [illegible] $d_\Sigma f \ldots (\rho^*)^{-1} d_{\mathbf{R}^{n-1}} \rho^* f$ [illegible] ρ_* [illegible] δ_Σ [illegible] $\delta_{\mathbf{R}^{n-1}}$ [illegible]

Proposition 3.6. *The following intertwining relations for exterior and interior differentiation operators hold.*

(1) *If* $U : D^\pm \to \wedge \mathbf{R}^n$ *and* $N_*(U), N_*(dU) \in L_2(\ldots)$, *then* $N^+U|_\Sigma \in D(d_\Sigma)$ *and* $(N^+U|_\Sigma) \ldots d_\Sigma(N^+U|_\Sigma) \ldots N^+(dU|_\Sigma)$. *If* $U : D^\pm \to \wedge \mathbf{R}^n$ *and* $N_*(U)$, $N_*(\delta U) \in L_2(\ldots)$, *then* $\nu \lrcorner U|_\Sigma \in D(\delta_\Sigma)$ *and* $(N^-U|_\Sigma) \ldots \nu \wedge \delta_\Sigma(\nu \lrcorner U|_\Sigma) \ldots N^-(\delta U|_\Sigma)$.

(2) *If* $h \in N^+L_2$, $d_\Sigma h \in L_2(\ldots)$ *and* $x \notin \Sigma$, *then*

$$d \int_\Sigma E(y - x)\nu(y)h(y)\,d\sigma(y) \ldots \int_\Sigma E(y - x)\nu(y)(d_\Sigma h)(y)\,d\sigma(y).$$

If $h \in N^-L_2$, $\delta_\Sigma(\nu \lrcorner h) \in L_2(\ldots)$ *and* $x \notin \Sigma$, *then*

$$d \int_\Sigma E(y - x)\nu(y)h(y)\,d\sigma(y) \ldots \int_\Sigma E(y - x)\nu(y)(\nu \wedge \delta_\Sigma(\nu \lrcorner h))(y)\,d\sigma(y).$$

[illegible] $N^+(dU|_\Sigma) \ldots d_\Sigma(N^+U|_\Sigma)$ [illegible] $i : \ldots \to \mathbf{R}^n$ [illegible] $N^+U|_\Sigma \ldots i^*(U)$ [illegible] $N^-(\delta U|_\Sigma) \ldots \nu \wedge \delta_\Sigma(\nu \lrcorner U|_\Sigma)$ [illegible]

[illegible]

Proposition 3.7. *The operator* [illegible] *relates to the splitting of* [illegible]$\mathbf{R}^n$ *into homogeneous* **k***-vectors as follows.*

[illegible] *If* **h** [illegible] **L**$_2$[illegible] *and* [illegible]$\mathbf{N}^+\mathbf{h}$[illegible]*, then* [illegible]$\mathbf{C}^{\pm}\mathbf{h}$ [illegible]$_{+2}$[illegible]*. If* **h** [illegible] **L**$_2$[illegible] *and* [illegible]$\mathbf{N}^-\mathbf{h}$[illegible]*, then* [illegible]$\mathbf{C}^{\pm}\mathbf{h}$ [illegible]$_{-2}$[illegible]*. Thus, if* **h** [illegible] **L**$_2$[illegible] *and* **h**[illegible]*, then* $\mathbf{C}^{\pm}\mathbf{h}$ *is a* **k***-vector field.*

[illegible] *For a monogenic field* **F** [illegible] $\sum^n_{=0}$ **F** [illegible] *in* $\mathbf{D}^{\pm}$*, i.e.* [illegible]**d** [illegible] δ **F** [illegible]*, where* **F** [illegible] $\mathbf{D}^{\pm} \to$ [illegible] $\mathbf{R}^n$*, the following are equivalent.*

- *All homogeneous parts* **F** [illegible] *of* **F** *are monogenic, i.e.* [illegible]**d** [illegible] δ **F** [illegible].
- **F** *is two-sided monogenic, i.e.* **DF**[illegible] **FD***, where* **FD**[illegible] $\sum^n_{=1}$[illegible]∂[illegible]**F**[illegible]**e**[illegible].
- **F** *satisfies* **dF**[illegible] δ**F**[illegible].

[illegible]

$$\int_\Sigma \mathbf{E}(\mathbf{y}-\mathbf{x}) \wedge \mathbf{v}(\mathbf{y}) \wedge \mathbf{h}(\mathbf{y})\, d\sigma(\mathbf{y}) \; [\text{illegible}] \int_\Sigma [\text{illegible}] (\mathbf{y}-\mathbf{x})\, \mathbf{v}(\mathbf{y}) \wedge [\text{illegible}]\, \mathbf{h}(\mathbf{y}) [\text{illegible}]\, d\sigma(\mathbf{y}), \qquad \mathbf{x} [\text{illegible}],$$

$$\int_\Sigma \mathbf{E}(\mathbf{y}-\mathbf{x}) \lrcorner [\text{illegible}] \mathbf{v}(\mathbf{y}) \lrcorner \mathbf{h}(\mathbf{y}) [\text{illegible}]\, d\sigma(\mathbf{y}) \; [\text{illegible}] - \int_\Sigma [\text{illegible}] (\mathbf{y}-\mathbf{x})\, \mathbf{v}(\mathbf{y}) \lrcorner [\text{illegible}]\, \mathbf{h}(\mathbf{y}) [\text{illegible}]\, d\sigma(\mathbf{y}), \qquad \mathbf{x} [\text{illegible}],$$

[illegible] **x** [illegible] **E**[illegible]**x** [illegible] **x** [illegible] ^{2} [illegible] R[illegible] N[illegible]

$$\mathbf{L}_2[\text{illegible}] \; [\text{illegible}] \; \mathbf{L}_2[\text{illegible}] \; [\text{illegible}] \; \mathbf{L}_2^D[\text{illegible}] \; [\text{illegible}] \; \mathbf{L}_2[\text{illegible}],$$

[illegible] $\mathbf{L}_2$[illegible] R[illegible] $\mathbf{L}_2$[illegible] N[illegible] $\mathbf{L}_2^D$[illegible] D[illegible] $\mathbf{L}_2$[illegible] $\mathbf{L}_2$[illegible] $\mathbf{L}_2^D$[illegible] $\mathbf{L}_2$[illegible] $\mathbf{L}_2^D$[illegible] [illegible]$\mathbf{f}$[illegible]^2_D[illegible] [illegible]$\mathbf{f}$[illegible]$^2_2$ [illegible] $\mathbf{f}$[illegible]$^2_2$ [illegible] $\mathbf{L}_2$[illegible]

$$[\text{illegible}]\mathbf{f}[\text{illegible}]^2 [\text{illegible}] \{[\text{illegible}]\, \mathbf{u}[\text{illegible}]^2_2 \; [\text{illegible}]\mathbf{u}[\text{illegible}]^2_2 [\text{illegible}]\; \mathbf{u} [\text{illegible}]\, D[\text{illegible}],\; \mathbf{u}[\text{illegible}]\; \mathbf{f}\},$$

[illegible] $\mathbf{L}_2$[illegible]

Lemma 3.8. *If* [illegible] *is an unbounded Lipschitz graph, then* $\mathbf{L}_2$[illegible] *is dense and not closed in* $\mathbf{L}_2$[illegible]*. If* [illegible] *is a bounded Lipschitz surface, then* $\mathbf{L}_2$[illegible] *is a closed subspace of* $\mathbf{L}_2$[illegible] *of finite codimension. In particular, if* $\mathbf{D}^+$ *is Lipschitz diffeomorphic to the unit ball, then the codimension is* [illegible] *and*

$$\mathbf{L}_2[\text{illegible}] \{\mathbf{f} \; [\text{illegible}] \; \mathbf{L}_2[\text{illegible}] \; \mathbf{f}_0[\text{illegible}] \; \mathbf{f}_n[\text{illegible}] \int_\Sigma \boldsymbol{\nu} \wedge \mathbf{f}_{n-1} d\sigma [\text{illegible}] \int_\Sigma \boldsymbol{\nu} \lrcorner \mathbf{f}_1 d\sigma [\text{illegible}]\},$$

where **f** [illegible] $\to$ [illegible] $\mathbf{R}^n$ *denotes the* **k***-vector part of* **f***.*

Proof. [illegible] $\mathbf{N}^+\mathbf{L}_2$ [illegible] $\mathbf{N}^-\mathbf{L}_2$ [illegible] **d** [illegible] δ [illegible] $\mathbf{R}^n$ [illegible]

$$\delta(\mathbf{F}\mathbf{e}_{\overline{n}}) [\text{illegible}] \mathbf{dF}[\text{illegible}]\mathbf{e}_{\overline{n}},$$

f, e_n ... f, e_n ... $f \in N^+L_2$ ... N^+L_2 ... N^-L_2 ... d_Σ ... □

... $E^\pm$ ... $N^\pm$ ... E ... $E^+ - E^-$ ... N ... $N^+ - N^-$ ... E^2 ... N^2 ... I ... E ...

$$Nf \quad \nu \hat{f} \nu$$

... *rotation operator*

$$ENf(x) \quad \int_\Sigma E(y - x) \hat{f}(y) \nu(y) d\sigma(y).$$

... EN ...

$$I \quad EN \quad E^+N^+ \quad E^-N^- , \qquad N I \quad EN N \quad N^+E^+ \quad N^-E^- ,$$
$$I - EN \quad E^+N^- \quad E^-N^+ , \qquad N I - EN N \quad N^+E^- \quad N^-E^+ .$$

... $I \quad EN$ ... $E^+ : N^+L_2 \to E^+L_2$ ... $E^- : N^-L_2 \to E^-L_2$ ... $I \pm EN$ ... L_2 ... $F^\pm : D^\pm \to \mathbb{R}^n$ ...

Theorem 3.9. *Let* ... *be a strongly Lipschitz surface. Then* $\lambda \quad EN : L_2 \to L_2$ *is a Fredholm operator with index zero for all* $\lambda \in \mathbf{R}$ *(and more generally in a double sector around the real axis).*

... L_2 ... L_2 ... L_2^D ...

Theorem 3.10. *All four projections* $E^\pm$ *and* $N^\pm$ *leave each of the subspaces* L_2, L_2 *and* L_2^D *invariant and act boundedly in them. All eight restricted projections*

$$N^+ : E^\pm L_2^x \longrightarrow N^+L_2^x, \qquad N^- : E^\pm L_2^x \longrightarrow N^-L_2^x,$$
$$E^+ : N^\pm L_2^x \longrightarrow E^+L_2^x, \qquad E^- : N^\pm L_2^x \longrightarrow E^-L_2^x$$

are Fredholm operators, for $x \quad R, N, D$. *All eight maps are injective when* $x \quad R$, *i.e. when acting in the range* L_2, *for all strongly Lipschitz surfaces* .

Proof of Theorem 3.10. ... $E^\pm$ ... $N^\pm$ ... $I \pm EN$ ... L_2^x ...

... L_2 ... L_2^D ...

[illegible] $I \pm E$ [illegible] L_2 [illegible] L_2^D [illegible] L_2 [illegible]

[illegible] $I \pm E$ [illegible] L_2 [illegible] L_2 [illegible]

$$\begin{array}{ccccccccc} ? & \longrightarrow & L_2(?) & \longrightarrow & L_2^D(?) & \xrightarrow{\Gamma} & L_2(?) & \longrightarrow & ? \\ & & \downarrow{\scriptstyle I\pm E} & & \downarrow{\scriptstyle I\pm E} & & \downarrow{\scriptstyle I\pm E} & & \\ ? & \longrightarrow & L_2(?) & \longrightarrow & L_2^D(?) & \xrightarrow{\Gamma} & L_2(?) & \longrightarrow & ?\,. \end{array}$$

[illegible] i [illegible] $\mathbb{N}$ [illegible] $\mathbb{R}$ [illegible] i [illegible] L_2

Then there are unique Cauchy type harmonic conjugates $V_1 : D \to \wedge^{k-2}\mathbf{R}^n$ *and* $V_2 : D \to \wedge^{k+2}\mathbf{R}^n$ *such that*

$$\|N_*(V_1)\|_2 + \|N_*(V_2)\|_2 \lesssim \|N_*(U)\|_2 + \|N_*(dU)\|_2 + \|N_*(\delta U)\|_2.$$

[illegible] L_p [illegible] k [illegible] $\leq k \leq n-$ [illegible] L_p [illegible] $p >$ [illegible] L_2 [illegible] L_2^D [illegible] dU [illegible] δU [illegible] U [illegible] k [illegible] L_2 [illegible] $N_*(U)$, $N_*(dU)$ [illegible] $N_*(\delta U)$ [illegible] L_2 [illegible] k [illegible] U [illegible]

$$h \xrightarrow{(C^\pm)_{k+2}} V_2, \qquad h \xrightarrow{(C^\pm)_k} U, \qquad h \xrightarrow{(C^\pm)_{k-2}} V_1.$$

[illegible] h [illegible] $C^\pm h$ [illegible] U

Corollary 4.2. *Let* $D^\pm$ [illegible] $\mathbf{R}^n$ *be a Lipschitz graph, interior or exterior domain, and let* $\leq k \leq n-$ [illegible]. *Then the range and null space of* $C^\pm$, *with domain* L_2^D [illegible], *are*

$$\mathcal{R}(C^\pm) = \{U : D^\pm \to \wedge \mathbf{R}^n ; \delta dU = d\delta U, N_*(U), N_*(\delta U), N_*(dU) \in L_2\},$$

$$\mathcal{N}(C^\pm) = \{F|_\Sigma ; F : D^\mp \to \wedge \mathbf{R}^n, dF = \delta F, N_*(F) \in L_2\},$$

with the same modifications of $\mathcal{R}(C^\pm)$ *as in Theorem 4.1 when* $D^\pm$ *is an exterior domain and when* $D^\pm$ *is an interior domain and* k [illegible] $, n-$ [illegible], *and where* $F \to$ [illegible] *when* $x \to \infty$ *when* $D^\mp$ *is an exterior domain and* $F \in \mathcal{N}(C^\pm)$. *The operator*

$$C^\pm : L_2^D [\text{illegible}] / \mathcal{N}(C^\pm) \longrightarrow \mathcal{R}(C^\pm)$$

is an isomorphism. Thus, if $U \in \mathcal{R}(C^\pm)$, *its Cauchy type harmonic conjugates are*

$$V_1 = (C^\pm (C^\pm)^{-1} U)_{k-2} \quad \text{and} \quad V_2 = (C^\pm (C^\pm)^{-1} U)_{k+2}.$$

Proof. [illegible] $U \in \mathcal{R}(C^\pm)$ [illegible] $U = C^\pm h$ [illegible] $h \in L_2^D$ [illegible] U [illegible] $N_*(U), N_*(\delta U), N_*(dU) \in L_2$ [illegible] $\delta dU = -\delta\delta(C^\pm h)_{+2}$ [illegible] U [illegible]

$F : D^{\mp} \to \mathbb{R}^n$ dF δF $N_* F \in L_2$ $F|_\Sigma$ $D^{\mp}$ $C^{\pm} F|_\Sigma$ $F|_\Sigma \in N C^{\pm}$ □

Remark 4.3. $E^{\pm}$ $E^{\pm}$ k $N^{\pm} L_2$ $F : D^{+} \to \mathbb{R}^n$ g f $F|_\Sigma$ $f \in E^{+} L_2$ $N^{-} f$ g f $E^{+} h$ $h \in N^{-} L_2$ $N^{-} E^{+} h$ g $N^{-} L_2$ $N^{-} E^{+}|_{-L_2}$ $N^{\pm} L_2$ $E^{\pm}|_{L_2(\Sigma;\wedge^k)}$ $N^{\pm} E^{\pm}|_{\pm L_2}$

Uniqueness proof of Theorem 4.1. $h \in L_2$ U $C^{\pm} h$ $h \in L_2$ $C^{\pm} h$ V_1 $C^{\pm} h$ $_{-2}$ V_2 $C^{\pm} h$ $_{+2}$

$$V_1(x) = C^{\pm}(N^{-} h)(x) = \pm \int_\Sigma E(y - x)\, \nu(y) \lrcorner\, h(y)\, d\sigma(y),$$

$$V_2(x) = C^{\pm}(N^{+} h)(x) = \pm \int_\Sigma E(y - x)\, \nu(y) \wedge h(y)\, d\sigma(y),$$

$x \in D^{\pm}$ V_1 V_1 $_{-2}$ V_2 V_2 $_{+2}$ V_1 V_2 U dV_2 d δV_2 $_{+3}$ δV_2 $-dV_2$ dV_1 d δV_1 $_{+1}$ V_2 V_2

$$E^{\pm} N^{+} h.$$

$E^{\pm} N^{-} h$

h_1 $N^{+} h$ $h_1 \in L_2^D$ $E^{\pm} h_1 = \frac{1}{2}(I \pm E) h_1 = \frac{1}{2}(I \pm EN) h_1$

$$\begin{array}{ccc} L_2^D & \xrightarrow{I \pm E} & L_2^D \\ \downarrow & & \downarrow \\ L_2 & \xrightarrow{I \pm E} & L_2, \end{array}$$

i $I \pm EN$ L_2 L_2^D $(I \pm EN) h_1 \in L_2^D$

h_1 L_2^D
$N^- h$ L_2^D
$E^\pm$ $N^+ h$ $E^\pm$ $N^- h$ $N^+ h$ $E^\mp L_2 \cap N^+ L_2$
$N^- h$ $E^\mp L_2 \cap N^- L_2$ $N^+ h$ $N^- h$
h L_2 V_1 V_2
$C^\pm h$ □

Existence proof of Theorem 4.1. U $D \to$ $\mathbf{R}^n$
δdU $d\delta U$ $N_* U$, $N_* dU$, $N_* \delta U$ L_2 D
$U, dU, \delta U \to$ $x \to \infty$ D
δU k dU k $n -$
V_1 $D \to$ ${}^{-2}\mathbf{R}^n$ V_2 $D \to$ ${}^{+2}\mathbf{R}^n$
h L_2^D $C^\pm h$ V_1 U V_2

$$N_* V_1 \quad N_* V_2 \quad h_D \lesssim N_* U \quad N_* dU \quad N_* \delta U .$$

k h_2 $N^+ L_2^D$
dU d $C^\pm h_2$

$$N^+ E^\pm h_2 \quad N^+ dU|_\Sigma .$$

dU
$N^+ dU|_\Sigma$ L_2

$$N^+ E^\pm \; N^+ L_2 \longrightarrow N^+ L_2$$

$E^\pm$ $N^+ L_2 \to E^\pm L_2$
N^+ $E^\pm L_2 \to N^+ L_2$

$$\lambda^2 - N^+ E^+ N^+ \quad N^- E^- N^+ \quad \lambda N^2 - N \quad E^2$$
$$\lambda \quad EN \; N \; \lambda - \quad - EN \; N,$$
$$\lambda^2 - N^+ E^- N^+ \quad N^- E^+ N^+ \quad \lambda N^2 - N - E^2$$
$$\lambda \quad - EN \; N \; \lambda - \quad EN \; N.$$

L_2 λ
f $N^+ L_2$
$N^- f$ $\lambda^2 - N^+ E^\pm$ $N^+ L_2 \to N^+ L_2$
λ
$N^+ E^\pm$ $N^+ L_2 \to N^+ L_2$
h_2 $N^+ L_2$

$$N^+ E^\pm h_2 - dU|_\Sigma ,$$

$dU|_\Sigma$ $E^\pm L_2$
$N^- L_2 \cap E^\pm L_2$ { }
D^+ $\leq k \leq n -$
$dU|_\Sigma$ $E^\pm L_2$ k $n -$ dU $E^\pm L_2$
k $n -$

$$\int_\Sigma \nu \wedge dU \, d\sigma \pm \int_{D^\pm} d \, dU \, dx$$

d^2 … D^- … $|dU| \lesssim |x|^{-n}$
$x \to \infty$ … $E^{\pm}h_2$ … $dU|_\Sigma$ … $E^{\pm}L_2 \cap N^-L_2$ … $\{ \}$

$$dU \quad d\, C^{\pm}h_2, \quad h_2 \in N^+L_2^D,$$

$$h_2 \quad h_2 \quad \|N_*U\|_2 \quad \|N_* dU\|_2 \gtrsim \|h_2\| \approx \|h_2\|_2 \quad \|h_2\|_2$$

$$\delta U \quad \delta\, C^{\pm}h_1, \quad h_1 \in N^-L_2^D,$$

$$\|h_1\|_2 \quad h_1\|_2 \lesssim \|N_*U\|_2 \quad \|N_*\delta U\|_2$$

… V_1 … V_2 … V … $C^{\pm}h$
i … j …

$$d\, U - V_1 \quad - V_2 \quad dU - \quad - dU,$$
$$\delta\, U - V_1 \quad - V_2 \quad \delta U - \delta U - ,$$

… $U - V_1 \quad - V_2 \,|_\Sigma \in E^{\pm}L_2$ …

$$h \quad U - V_1 \quad - V_2 \,|_\Sigma \quad h_1 \quad h_2 \in L_2^D,$$

V_1 … V_1 … -2 … V_2 … V_2 … $+2$ …

$$C^{\pm}h \quad U - V_1 \quad - V_2 \quad V_1 \quad V_2 \quad V_1 \quad U \quad V_2.$$

… $\|N_*V_1\|_2 \quad \|N_*V_2\|_2 \lesssim \|N_*U\|_2 \quad \|h_1\|_2 \quad \|h_2\|_2 \lesssim \|N_*U\|_2 \quad \|N_* dU\|_2$
$\|N_*\delta U\|_2$ … $\|h\|_D \lesssim \| U - V_1 \quad - V_2 \,|_\Sigma\|_2 \quad \|h_1\|_D \quad \|h_2\|_D \lesssim \|N_*U\|_2$
$\|N_* dU\|_2 \quad \|N_*\delta U\|_2$ … □

5. Other types of harmonic conjugates

… U … $D \to \mathbf{R}^n$ …
… k … V_1 … $D \to \;{}^{-2}\mathbf{R}^n$ … V_2 … $D \to \;{}^{+2}\mathbf{R}^n$ …
… U … d … δ … V_1 … U … V_2 …
… V_1 … V_2 … V …
…
… V …
…

Proposition 5.1. *Assume that* V_1 *and* V_2 *are harmonic conjugates to* U *in* D … D^+, *with* $N_*U, N_*V_1, N_*V_2 \in L_2$ … . *Then they are of Cauchy type if and only if there exists* U^- … $D^- \to \mathbf{R}^n$ *with harmonic conjugates* V_1^- … $D^- \to \;{}^{-2}\mathbf{R}^n$ *and* V_2^- … $D^- \to \;{}^{+2}\mathbf{R}^n$ *in* D^-, *with* $N_*U^-, N_*V_1^-, N_*V_2^- \in L_2$ … *(and decay at infinity when* D^- *is an exterior domain), such that*

$$V_1|_\Sigma \quad V_1^-|_\Sigma \qquad \text{and} \qquad V_2|_\Sigma \quad V_2^-|_\Sigma .$$

Proof. … V … h … $\to \mathbf{R}^n$
C^+h … $V_1 \quad U \quad V_2$ … $V_1^- \quad U^- \quad V_2^-$ … C^-h …
… $E^+h \quad E^-h$ … h … $V_1^- \quad U^- \quad V_2^-$ …
… h … $U|_\Sigma \quad U^-|_\Sigma$ … $V_1 \quad U \quad V_2\,|_\Sigma$ … $V_1^- \quad U^- \quad V_2^-\,|_\Sigma$ … $\to \mathbf{R}^n$
… C^+h … $V_1 \quad U \quad V_2$ □

$\mathbf{D}^+$ $\mathbf{D}^-$

transmission problem

$\mathbf{D}^\pm$

$\mathbf{D}$

$\mathbf{D}^*$

$\mathbf{d}$ $\boldsymbol{\delta}$

$\int_{\mathbf{R}^n}$

k , n −

N T ∩ L_2 D

Definition 5.3. U D$^\pm$ → R^n δdU dδU V_1 D → $^{-2}$R^n V_2 D → $^{+2}$R^n *Hodge type harmonic conjugates* U d δ V_1 U V_2 ν ⌟ $V_1|_\Sigma$ ν ∧ $V_2|_\Sigma$

Theorem 5.4. *Assume that* D R^n *is Lipschitz diffeomorphic to the unit ball, let* ≤ k ≤ n *and let* U L_2 D *be such that* dU, δU L_2 D *and* δdU dδU. *If* k n − *, assume that* dU *, and if* k *assume that* δU *. Then there exists Hodge type harmonic conjugates* V_1, V_2 *to* U *in* D *such that* $\|V_1\|_{L_2(D)} \lesssim$ $\|\delta U\|_{L_2(D)}$ *and* $\|V_2\|_{L_2(D)} \lesssim \|dU\|_{L_2(D)}$. *The conjugates are unique, except if* k *, when* V_1 *is unique modulo constants, and if* k n − *, when* V_2 *is unique modulo constants.*

Proof. F δU F dU □

Remark 5.5. n k

V V_2 D → ^{2}R^2 ν ∧ V $|_\Sigma$

D n ≥ k

$$V x = \int_\Sigma E(y - x) \wedge \nu(y) \; h(y)\, d\sigma(y), \qquad h - \to {}^0\mathbf{R}^n$$

ν ∧ V $|_\Sigma$ ≤ k ≤ n D ν(y) y

$$V_2(x) = \int_\Sigma E(y - x) \wedge \nu(y) \wedge h(y)\, d\sigma(y) - x \wedge \int_\Sigma \frac{y \wedge h(y)\, d\sigma(y)}{\sigma_{n-1}|y - x|^n}$$

V_1

k , n −

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