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Boundary derivatives of the phases of inner and outer functions and applications

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SUMMARY

We prove that boundary derivatives of the phases of inner functions exist and are positive almost everywhere, but those of outer functions, on the other hand, have zero mean on the boundary. The concepts and results have definitive applications to the definitions of instantaneous frequency and mono-components complying with requirements in physics and contemporary study of analytic signals. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: inner function; outer function; Blaschke product; singular inner function; non-tangential boundary limit; angular limit; angular derivative; amplitude–phase modulation; analytic signal; instantaneous frequency

1. INTRODUCTION

Denote by $\mathbb{D}$ the open unit disc in the complex plane, and

$$\tau_a(z)=\frac{z-a}{1-\overline{a}z}, \quad |a|<1 \tag{1}$$

As Möbius transform, $\tau_a\in\mathrm{Aut}(\mathbb{D})$. The mapping has analytic extension to $\mathbf{C}\setminus\{1/\overline{a}\}$. In particular, it maps $\partial\mathbb{D}$ onto $\partial\mathbb{D}$ in one-to-one manner, and it preserves the orientation. Since $\tau_a(\mathrm{e}^{\mathrm{i}t})$ is unimodular, for a strictly monotone function $\theta_a(t)$ depending on a, we have

$$\tau_a(\mathrm{e}^{\mathrm{i}t})=\mathrm{e}^{\mathrm{i}\theta_a(t)}$$

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Simple computation shows that the derivative of the phase function $\theta_a(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\theta_a(t)}{2\pi}=\frac{1}{2\pi}\frac{1-|a|^2}{|\mathrm{e}^{\mathrm{i}\phi}-a|^2}=\frac{1}{2\pi}\frac{1-|a|^2}{1-2|a|\cos(t_a-t)+|a|^2}>0$$

where $a=|a|\mathrm{e}^{\mathrm{i}t_a}$, and the last expression is the Poisson kernel on the circle at a (see [1]). This positive-phase derivative property can be immediately extended to finite Blaschke products. In fact, the phase function of a finite Blaschke product is the finite sum of the phase functions of the composing Möbius transforms (see, for instance, [2]).

The phenomenon of positive-phase derivative property may also be seen in the simplest singular inner functions case. Recall that a singular (or singular inner) function is given by the formula

$$S(z)=\mathrm{e}^{-\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\,\mathrm{d}\mu(t)}$$

where $\mathrm{d}\mu$ is a positive regular Borel measure singular (orthogonal) to Lebesgue measure. The simplest singular inner function is given by the point mass at a single point, say at $t=0$, that is $\mathrm{d}\mu(t)=\delta_0(t)\mathrm{d}t$, where δ_0 is the usual Dirac function. We have, in the case,

$$S(z)=\mathrm{e}^{(z+1)/(z-1)}$$

For $z=\mathrm{e}^{\mathrm{i}t}$, we have

$$S(\mathrm{e}^{\mathrm{i}t})=\mathrm{e}^{\mathrm{i}\theta(t)},\quad \theta(t)=-\cot\frac{t}{2}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t)=\frac{1}{2}\frac{1}{\sin^2\frac{t}{2}}>0,\quad t\neq 0$$

If $\mathrm{d}\mu$ is a sum of a finite many of point masses, we have the same conclusion $\theta'(t)>0$ for all t's but those at which the point masses are placed. This phenomenon is also observed in [3].

Functions having this property are not necessarily to be unimodular. A family that possesses global non-negative phase derivatives is the family of starlike mappings. Let Ω be a simply connected open and starlike domain containing the point $z=0$ and bounded by a Jordan curve. Let $f:\mathbb{D}\to\Omega$ be a conformal mapping with $f(0)=0$. By a famous theorem of Carathéodory (see [4]) the mapping f has a one-to-one continuous extension from $\overline{\mathbb{D}}\to\overline{\Omega}$. Then f is a starlike mapping about zero if and only if $f'(0)\neq 0$, and

$$Re[zf'(z)/f(z)]>0\quad \text{for } z\in\mathbb{D}$$

(see, for instance, [5]). The mapping $f:\overline{\mathbb{D}}\to\overline{\Omega}$ induces a natural parameterization for the boundary

$$f(\mathrm{e}^{\mathrm{i}t})=\rho(t)\mathrm{e}^{\mathrm{i}\theta(t)},\quad \rho(t)>0$$

If both f and f' are in the Hardy $H^1(\mathbb{D})$ space, then $f(\mathrm{e}^{\mathrm{i}t})$ is of bounded variation, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t)\geqslant 0\quad \text{a.e. } t\in[0,2\pi)$$

(see, for instance, [1, p. 93, 8(b)]).

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In this note, we restrict ourself to the function classes involved in the Nevanlinna canonical factorization theorem: A function f is in the Nevanlinna class, $\mathcal{N}$, if and only if f is of the form

$$f = CFBS_1/S_2, \quad |C| = 1$$

where B is a Blaschke product, S_1 and S_2 are singular functions, F is an outer function and C is a constant. Except for the choice of the constant C, the decomposition is unique. Functions in the Nevanlinna class with $S_2 = 1$ constitute a class $\mathcal{N}^+$. A function f is in the Hardy $H^p(\mathbb{D})$, $0 < p \leqslant \infty$, if and only if $f \in \mathcal{N}^+$ and its outer function part, F, is in $H^p(\mathbb{D})$.

A function f is an outer function in $\mathbb{D}$ if and only if f has the form

$$f(z) = C e^{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log h(e^{it})\, dt}, \quad |C| = 1 \tag{2}$$

where h is a positive measurable function and $\log h \in L^1(\partial\mathbb{D})$, and C is a constant. A function f defined in (2) is in Hardy H^p, $0 < p \leqslant \infty$, if and only if $h \in L^p$. A function f is an inner function in $\mathbb{D}$ if and only if $f = CBS$, where C is a constant, $|C| = 1$, B is a Blaschke product and S is a singular function, or, equivalently, f maps $\mathbb{D}$ into $\mathbb{D}$ with unimodular non-tangential boundary values almost everywhere on $\partial\mathbb{D}$ (see, for instance, [1]). There are parallel notions and results for the upper-half complex plane.

A natural question is whether there is a similar boundary behavior for Blaschke products of infinite zeros, and for singular functions constructed from general singular measures on $\partial\mathbb{D}$, and what is for outer functions? The mentioned functions are all in the Nevanlinna class and therefore have *non-tangential boundary values* (or *angular limits*). The concept of the phase derivative and monotonicity of the phase function all depend on suitable parameterizations of the boundary curve. As example, for the boundary values of a Blaschke product there exists a real-valued function $\theta(t)$ such that

$$B(e^{it}) = e^{i\theta(t)} \quad \text{a.e.}$$

The choice for the function $\theta(t)$, however, is not unique. In fact,

$$B(e^{it}) = e^{i(\theta(t)+2\pi k(t))} \quad \text{a.e.}$$

for any function $k : [0, 2\pi) \to \mathbf{Z}$, where $\mathbf{Z}$ denotes the set of all integers.

The above questions have root in contemporary study of theoretical signal analysis. They motivated a number of researchers who have subsequently worked out partial results in this direction (see [2, 3, 6–11]). But none of the existing results are able to concern Blaschke products of infinite zeros or singular inner functions from general positive Borel measures singular to Lebesgue measure, or for general inner functions and outer functions. Very relevant results, such as Julia–Wolff–Carathéodory's theorem, however, already exist in complex analysis of one variable. What we do in this note is to interpret the existing results and provide a formulation in the right context under which the above questions are answered. We show that under our formulation, the phase derivatives of inner functions exist and are positive almost everywhere. For outer functions, it is different. The simplest case of outer functions is studied in [3]. In strict mathematical formulation and by using a different approach, we show that under a mild condition phase derivatives of outer functions exist almost everywhere and are of zero mean on the boundary. In Section 2 we develop the theory and, in Section 3, as application, we interpret the obtained results in the signal analysis context.

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2. BOUNDARY DERIVATIVE OF PHASE FUNCTIONS

For an analytic function $f:\mathbb{D}\to\mathbf{C}$, writing $f(re^{it})=\rho_r(t)e^{i\theta_r(t)}, r<1$, and taking derivative to both sides with respect to t, we obtain that

$$\frac{d}{dt}\theta_r(t)=Re\left(z\frac{f'(z)}{f(z)}\right) \tag{3}$$

Note that this relation may be extended to the points re^{it} for $r\geqslant 1$ if the function f has analytic continuation across an open interval containing the point re^{it}. For general points $z=e^{it}$, there is a substitution called *angular derivative* defined as follows.

Definition 2.1
Let f be analytic in the annulus region $\{z\in\mathbf{C}|r<|z|<1\}, 0<r<1$. Suppose that for some Stolz angle $\Delta_{\alpha_0}(\zeta)$ at $\zeta\in\partial\mathbb{D}$ (see [5, p. 6]),

$$\Delta_{\alpha_0}(\zeta)=\{z\in\mathbb{D}||\arg(1-\overline{\zeta}z)|<\alpha_0, |z-\zeta|<\rho\},\quad 0<\alpha_0<\pi/2,\ \ \rho<2\cos\alpha_0$$

the limit

$$\lim_{z\to\zeta, z\in\Delta_{\alpha_0}(\zeta)} f(z)=\sigma$$

exists, and for all the Stolz angles Δ_α at ζ for $\alpha_0<\alpha<\pi/2$ the limits exist and are of the same value σ. In the case we denote the limit by

$$\lim_{S:z\to\zeta} f(z)=\sigma$$

and call it *the angular limit of f at ζ*. In this circumstance, this value σ is denoted by $f(\zeta)$. Note that here we allow the limit to be ∞ or $\pm\infty$. We say that f has the *angular derivative* $f'(\zeta)$ at $\zeta\in\partial\mathbb{D}$, if $f(\zeta)=\lim_{S:z\to\zeta} f(z)\neq\infty$ exists and if

$$\lim_{S:z\to\zeta}\frac{f(z)-f(\zeta)}{z-\zeta}=f'(\zeta)$$

Again, it allows the infinite values. In the sequel without further notice, the notation $f'(\zeta)$ itself indicates the existence and represents the value of the angular derivative of f at $\zeta\in\partial\mathbb{D}$.

It has been proved that the analytic function f has a finite angular derivative if and only if $f(z)$ has the finite angular limit $f'(\zeta)$ at $\zeta\in\partial\mathbb{D}$ [5, p. 79]. We are now ready to define the *boundary derivative of the phase function* of an analytic function $f:\mathbb{D}\to\mathbf{C}$.

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Definition 2.2
Let $f:\mathbb{D}\to\mathbf{C}$ be analytic, and $\zeta\in\partial\mathbb{D}$. If the angular limit $Re(zf'(z)/f(z))$ exists, then we denote it by

$$Df(\zeta)=\lim_{S:z\to\zeta} Re\left(z\frac{f'(z)}{f(z)}\right)$$

and call it the *boundary derivative of the phase*, or *phase derivative*, of f at ζ. Note that for a given ζ, $Df(\zeta)$ may not exist, and when exists, it may happen $\mathrm{D}f(\zeta)=\pm\infty$, or $\mathrm{D}f(\zeta)=\infty$. In the sequel the notation $Df(\zeta)$ itself indicates the existence, as well as represents the value of the phase derivative of f at $\zeta\in\partial\mathbb{D}$.

If f has a non-zero angular limit and a finite angular derivative $f'(\zeta)$ at $\zeta\in\partial\mathbb{D}$, then $Df(\zeta)=Re(\zeta f'(\zeta)/f(\zeta))$. If f has analytic extension across an open interval on $\partial\mathbb{D}$ containing ζ where $f(\zeta)\neq 0$, then $Df(\zeta)=Re(\zeta f'(\zeta)/f(\zeta))$. In this case, the observation made at the beginning of this section concludes that $Df(\zeta)=Re(\zeta f'(\zeta)/f(\zeta))=\theta'(t_\zeta)$, where $\zeta=\mathrm{e}^{\mathrm{i}t_\zeta}$. This gives the reason of the terminology 'boundary derivative of the phase' or 'phase derivative' in Definition 2.2.

We recall Julia–Wolff–Carathéodory's theorem [12, 13]:

Theorem 2.1 (*Julia–Wolff–Carathéodory*)
Let f be analytic, $f:\mathbb{D}\to\mathbb{D}$ and $\sigma,\zeta\in\partial\mathbb{D}$. Then

$$\lim_{S:z\to\zeta}\frac{\sigma-f(z)}{\zeta-z}=\sigma\bar{\zeta}\beta_f(\zeta,\sigma)$$

where

$$\beta_f(\zeta,\sigma)=\sup_{z\in\mathbb{D}}\left[\frac{|\sigma-f(z)|^2}{1-|f(z)|^2}\Big/\frac{|\zeta-z|^2}{1-|z|^2}\right]$$

If $\beta_f(\zeta,\sigma)$ is finite, then

$$\lim_{S:z\to\zeta} f(z)=\sigma \quad\text{and}\quad \lim_{S:z\to\zeta} f'(z)=\sigma\bar{\zeta}\beta_f(\zeta,\sigma)$$

The above may be re-formulated in the following form that is more pertinent to our purpose (see [5, p. 82]):

Theorem 2.2
Let f be analytic in $\mathbb{D}$ with an angular limit $f(\zeta)$ at $\zeta\in\partial\mathbb{D}$. If

$$f(\mathbb{D})\subset\mathbb{D},\quad f(\zeta)\in\partial\mathbb{D}$$

then the angular derivative $f'(\zeta)$ exists, and

$$0<\zeta\frac{f'(\zeta)}{f(\zeta)}=\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{|\zeta-z|^2}\,\frac{|f(\zeta)-f(z)|^2}{1-|f(z)|^2}\leqslant+\infty$$

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We have the following:

Theorem 2.3
If f is an inner function, then $Df > 0$ a.e. Moreover, if f has an analytic extension across an open interval containing $\zeta = e^{it_\zeta}$, then with the angular parameterization $f(e^{it}) = e^{i\theta(t)}$ the phase function $\theta(t)$ is differentiable at $t = t_\zeta$, and $0 < \theta'(t_\zeta) < +\infty$.

Proof
As inner function f has unimodular non-tangential boundary limits at almost all points on the unit circle, Theorem 2.2 can be directly used to conclude the theorem. □

Theorem 2.3, together with Theorems 6.1 and 6.2 of Chapter II, Section 6 of [1], then implies

Corollary 2.4
If B is a Blaschke product, then $DB > 0$ a.e. Let $B(z)$ be the Blaschke product with zeros $\{z_n\}$, and let $E \subset \partial\mathbb{D}$ be the set of accumulation points of $\{z_n\}$. Then $B(z)$ has analytic extension across each arc of $\partial\mathbb{D} \setminus E$. In particular, on each of those arcs the finite boundary derivatives of the phase function $\theta(t)$ of $B(z)$ exist and are positive.

Corollary 2.5
If S is a singular function, then $DS > 0$ a.e. Let $S(z)$ be the singular function determined by the measure μ on $\partial\mathbb{D}$, and let E be the closed support of μ. Then $S(z)$ has analytic extension across each arc of $\partial\mathbb{D} \setminus E$. In particular, on each of those arcs the finite boundary derivatives of the phase function $\theta(t)$ of $S(z)$ exist and are positive.

Example
We construct in below an example for an inner function that has $Df = +\infty$ almost everywhere on the unit circle $\partial\mathbb{D}$. Let $\{a_n\}_{n=1}^{\infty}$ be an interpolating sequence in $\mathbb{D}$, that is, for any sequence $\{b_n\} \in l^\infty$ there exist solutions $f \in H^\infty(\mathbb{D})$ such that $f(a_n) = b_n$ [1, 14]. We can further assume $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$ (i.e. it is additionally also a Blaschke sequence) that is a sufficient and necessary condition for the solution $f \in H^\infty$ and $f(a_n) = b_n$ is not unique, and in the case there exists an inner function f solving the interpolation problem [14, pp. 6, 62]. It suffices, therefore, to construct a *uniform interpolating sequence* [14, p. 63]. For our special purpose, we proceed as follows. We, along each radius $re^{ik\pi/3}, 0 < r < 1, k = 0, 1, \ldots, 5$, construct a uniform interpolating sequence, the availability of which is based on, for instance, Theorem 7.4, [14, p. 65]. We then combine the six interpolating sequences together to form a sum-interpolating sequence $\{a_n\}$. Later, we select a sequence $\{b_n\} \subset \mathbb{D}$ such that for each n, b_n is on the same radius as a_n, $|a_n| < |b_n| < 1$, and

$$\lim_{n\to\infty} \frac{1 - |a_n|}{1 - |b_n|} = \infty$$

Let now f be an inner function that solves the interpolating problem $f(a_n) = b_n, n = 1, 2, \ldots,$ where $\{a_n\}$ and $\{b_n\}$ are constructed as above. Let ζ be a point on $\partial\mathbb{D}$ such that $f(\zeta)$ exists as angular limit

$$\lim_{S: z\to\zeta} f(z) = f(\zeta) \quad \text{and} \quad |f(\zeta)| = 1$$

Such ζ distributes on $\partial\mathbb{D}$ almost everywhere. Let the points ζ and $f(\zeta)$ be situated on some half circle {e the half disc having the half circle as part of its boundary. Thus, the radius $R_{k_0}=\{r\mathrm{e}^{\mathrm{i}k_0\pi/3}|0<r<1\}$ is separated from both ζ and $f(\zeta)$ by a positive distance. On the basis of Theorem 2.2, we have

$$\begin{aligned}\zeta\frac{f'(\zeta)}{f(\zeta)} &\geqslant \sup_{a_n\in R_{k_0}} \frac{1-|z|^2}{|\zeta-z|^2}\frac{|f(\zeta)-f(z)|^2}{1-|f(z)|^2}\\ &= \sup_{a_n\in R_{k_0}} \frac{(1+|a_n|)|f(\zeta)-b_n|^2}{|\zeta-a_n|^2(1+|b_n|)}\frac{1-|a_n|}{1-|b_n|}\\ &=\infty\end{aligned}$$

Next we study outer functions and prove

Theorem 2.6
Let f be an outer function in some H^p space for $0<p\leqslant\infty$, and the analytic function f'/f in $\mathbb{D}$ belongs to the Hardy $H^1(\mathbb{D})$ space. Then the angular derivatives $f'(\mathrm{e}^{\mathrm{i}t})$ exist and are finite almost everywhere, and the function

$$\mathrm{e}^{\mathrm{i}t}\frac{f'(\mathrm{e}^{\mathrm{i}t})}{f(\mathrm{e}^{\mathrm{i}t})}$$

is integrable with

$$\int_0^{2\pi}\mathrm{e}^{\mathrm{i}t}\frac{f'(\mathrm{e}^{\mathrm{i}t})}{f(\mathrm{e}^{\mathrm{i}t})}\,\mathrm{d}t=0 \tag{4}$$

Proof
Since $f\in H^p$, the non-tangential boundary values of f are non-zero almost everywhere [1, p. 65, Corollary 4.2]. The existence of the finite non-tangential boundary limits of the function f'/f then implies that $f'(\zeta)$ exist and are finite almost everywhere. As the boundary value of a function in the Hardy H^1 space we have

$$\int_{\partial\mathbb{D}}\frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta=0$$

After changing to the angular parameter, the above relation becomes (4). □

Corollary 2.7
Under the assumptions of Theorem 2.6 there holds

$$\int_0^{2\pi} Df(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t=0 \tag{5}$$

Proof
The integrability of $\mathrm{e}^{\mathrm{i}t}f'(\mathrm{e}^{\mathrm{i}t})/f(\mathrm{e}^{\mathrm{i}t})$ implies that of $Df(\mathrm{e}^{\mathrm{i}t})=Re[\mathrm{e}^{\mathrm{i}t}f'(\mathrm{e}^{\mathrm{i}t})/f(\mathrm{e}^{\mathrm{i}t})]$. The zero mean assertion follows from (4). □

Corollary 2.7 shows that for a large class of outer functions though the boundary derivatives of the phase functions exist and are finite almost everywhere, they must be sometimes positive and sometimes negative so that they have zero mean.

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3. APPLICATIONS IN SIGNAL ANALYSIS

There are some recent studies on adaptive decomposition of signals by using mono-components. The related concepts are defined as follows.

Definition 3.1
Let F be a real-valued signal in $L^p([0,2\pi]), 1\leqslant p\leqslant\infty$. If we denote $F(t)=f(\mathrm{e}^{\mathrm{i}t})$, this is equivalent to let f be a real-valued signal in $L^p(\partial\mathbb{D}), 1\leqslant p\leqslant\infty$. The complex-valued signal

$$\tilde{A}f(\mathrm{e}^{\mathrm{i}t})=f(\mathrm{e}^{\mathrm{i}t})+\mathrm{i}\tilde{H}f(\mathrm{e}^{\mathrm{i}t})$$

is called the *analytic signal associated with* $f(\mathrm{e}^{\mathrm{i}t})$, where $\tilde{H}$ is the circular Hilbert transformation (see [1, 7]). The complex-valued signal Af has the *amplitude–phase modulation*

$$\tilde{A}f(\mathrm{e}^{\mathrm{i}t})=\rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$$

where $\rho(t)=\sqrt{f^2(\mathrm{e}^{\mathrm{i}t})+\tilde{H}f^2(\mathrm{e}^{\mathrm{i}t})}, \theta(t)=\arccos f(\mathrm{e}^{\mathrm{i}t})/\rho(t)$. The induced modulation

$$f(\mathrm{e}^{\mathrm{i}t})=\rho(t)\cos\theta(t)$$

is called the *analytic amplitude–phase modulation* (*analytic modulation*) of $f(\mathrm{e}^{\mathrm{i}t})$, and the function $\theta(t)$ is called the *analytic phase* of $f(\mathrm{e}^{\mathrm{i}t})$. In the sense given in Definition 2.2, the boundary derivative of the phase function of $\tilde{A}f$, $D(\tilde{A}f)$, if exists, is called the *analytic phase derivative* of $f(\mathrm{e}^{\mathrm{i}t})$. A real-valued such signal F in $L^p([0,2\pi])$ or f in $L^p(\partial\mathbb{D})$ is said to have *instantaneous frequency* if and only if $D(\tilde{A}f)$ exist and $D(\tilde{A}f)\geqslant 0$, a.e. on $\partial\mathbb{D}$, allowing the value $+\infty$. We say that F is a *mono-component* on $[0,2\pi]$, or f is a *mono-component* on $\partial\mathbb{D}$ if and only if f has *instantaneous frequency* on $\partial\mathbb{D}$.

Note that a function f is a mono-component if and only if its analytic phase derivative is of positive values, allowing $+\infty$, almost everywhere. And if and only if in this case, we say that the signal f has instantaneous frequency. We wish to stress that mono-component and instantaneous frequency are for the whole function: They are not a local or point-wise property.

Amplitude–phase modulation for a real-valued signal is not unique. However, it is the analytic modulation based on which the concept mono-component is defined. Analytic signals associated with real-valued signals in L^p spaces are boundary values of functions in the corresponding complex Hardy spaces [2]. They are interpreted as 'physically realizable' signals. The requirement $\theta'(t)\geqslant 0$, a.e. for analytic phase derivative is essential. Frequency in physics is defined to be the time of vibrations in the unit time interval, and, thus has to be non-negative. Quantitative signal analysis is crucially based on the positivity of frequencies. The concept instantaneous frequency has some controversies, however. Some authors call any analytic phase derivative $\theta'(t)$ instantaneous frequency without the requirement $\theta'(t)\geqslant 0$, a.e.

Associated with this notion one seeks adaptive decomposition of signals. Given an arbitrary signal f on $\partial\mathbb{D}$, one wishes to have the decomposition of f into mono-components in the fastest manner:

$$f(t)=\sum_1^\infty \rho_k(t)\cos\theta_k(t) \tag{6}$$

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where for each k the analytic modulation $\rho_k(t)\cos\theta_k(t)$ is a mono-component. A characterization of analytic modulation is

$$\tilde{H}(\rho_k(\cdot)\cos\theta_k(\cdot))(t)=\rho_k(t)\sin\theta_k(t)$$

Note that this is a generalization of the Fourier series theory to both contexts the unit circle and the real line. In particular, there hold

$$\tilde{H}(\cos k(\cdot))(t)=\sin kt, \quad k=1,2,\ldots$$

The notion 'fastest decomposition' depends on a metric of the space of signals. After a metric is assigned to a class of signals, there are different ways to formulate what is meant by 'fastest decomposition'. For instance, for a fixed $\varepsilon>0$, one can require to find the least index k_0 for which there exists a set of mono-components $\rho_k(t)\cos\theta_k(t), 1\leqslant k\leqslant k_0$, such that

$$\mathrm{dist}\left(f, \ {}^{k_0}\right.$$

On the boundaries,

$$e^{it} = \frac{i-s}{i+s}, \quad \kappa((-\infty,\infty]) = \{e^{it} \mid -\pi < t \leqslant \pi\}, \quad s = i\frac{1-e^{it}}{1+e^{it}}$$

and

$$s = \tan\frac{t}{2}, \quad t = 2\arctan s, \quad \frac{dt}{ds} = \frac{2}{1+s^2}, \quad \frac{ds}{dt} = \frac{1}{2}\sec^2\frac{t}{2}$$

The mono-components on the line are a subclass of the real-valued functions f on the line whose Hilbert transform Hf can be defined. This will allow the possibility of having f as the real part of the boundary value of a good analytic function in the upper-half complex plane. We define, in the same pattern as for signals on the unit circle, the *associated analytic signal* $A(f) = f + iHf$. By using the natural amplitude–phase modulation $A(f)(s) = \rho(s)e^{i\theta(s)}$, we can define the *analytic modulation* $f(s) = \rho(s)\cos\theta(s)$. The mono-components should be defined to be those for which the corresponding phase function θ, obtained via analytic modulation, has non-negative derivatives $\theta'(s)$ almost everywhere, allowing the $+\infty$ value. To make it a wider sense and avoid boundary parameterization paradox, we also approach it from the inner points of the domain. We proceed by converting the case to the already established theory in the disc, that is to map everything in the upper-half complex plane by Cayley transformation into the unit disc. The Cayley transformation preserves complex analyticity and is of monotonicity restricted on the boundaries. We call f a *mono-component on the line* if and only if $f(\kappa^{-1}(e^{it}))$ is a mono-component on the circle, and *the boundary derivative of the phase*, or *the phase derivative*, of f is defined to be $D_{\mathbb{R}}f(s) = D_{\mathbb{D}}(f\circ\kappa^{-1})(\kappa(s))dt/ds$, where $D_{\mathbb{D}}$ is identical with D in Definition 2.2. By abuse of notation we express

$$(\kappa^{-1}f)(w) = f(\kappa^{-1}w), \quad (\kappa f)(z) = f(\kappa z)$$

and, consequently, the above-defined phase derivative may be re-written as

$$D_{\mathbb{R}}f(s) = \frac{2}{1+s^2}(\kappa D_{\mathbb{D}}\kappa^{-1})f(s)$$

We indicate that there also holds

$$H = \kappa\tilde{H}\kappa^{-1}$$

In fact, if F is defined on $\mathbb{R}$, then we have $\kappa^{-1}HF = \tilde{H}\kappa^{-1}F$. This is because the both sides are the boundary value of the harmonic conjugate, with certain normalization, of the harmonic extension of $\kappa^{-1}F$ into the unit disc.

Theorem 3.2
The real parts of the boundary values of the inner functions on the line are all mono-components.

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