Complex Variables and Elliptic Equations
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Co-dimension-*p* Shannon sampling theorems

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e (e e) e e .

Keywords e

AMS Subject Classifications 2 0 , 30 0 , 30 10, 42 3

1. Introduction

e e e **R** e e

$$(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt} \quad = \frac{(\pi x)}{\pi x}.$$

e e e e e **C**, .e.,

$$(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{izt} \quad = \frac{(\pi z)}{\pi z}. \tag{1}$$

e e e 1].
e A, e χ_A e e e e A. $\mathbf{R}^m$
e e

$$(\underline{x}) = (\chi_{[-\pi,\pi]^m})\check{}(\underline{x}) = \frac{1}{(2\pi)^m}\int_{\mathbf{R}^m} e^{i\langle \underline{x},\underline{\xi}\rangle}\chi_{[-\pi,\pi]^m}(\underline{\xi})\ \underline{\xi}$$
$$= \prod_{i=1}^{m} \quad (x_i) = \prod_{i=1}^{m}\frac{(\pi x_i)}{\pi x_i}.$$

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Complex Variables and Elliptic Equations
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e e e e (1) e - e e ,
Q e e e e $\mathbf{R}^m$ $m+1$- e e e $\mathbf{R}_1^m$
e e 2]. e e e e e e
(e *inhomogeneous co-dimension*-1 *sinc function*) e e e e e
e e $e(x, \underline{\xi})$ $\mathbf{R}_1^m \times \mathbf{R}^m$, e e e e e
$e^{i\langle \underline{x}, \underline{\xi}\rangle}$ $\mathbf{R}^m \times \mathbf{R}^m$. e e, e e e - e -1
, e e e e e 2] e e
e e e ( ) e e $\mathbf{R}_1^m$ 3]. e, e e e - e -*p*
e e e e e
e e e 4].

2. Preliminaries

e e e e e e e e e
e e e].
e $\mathbf{e}_1, \ldots, \mathbf{e}_m$ e *basic elements* $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, e e $\delta_{ij} = 1$ $i = j$
$\delta_{ij} = 0$ e e, $i, j = 1, 2, \ldots, m$. e

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R},\ j = 1, 2, \ldots, m\},$$

$$\mathbf{R}_1^m = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R},\ \underline{x} \in \mathbf{R}^m\}.$$

$\mathbf{R}^m$ $\mathbf{R}_1^m$ e e , e e e , e *homogeneous* *inhomogeneous* e
e .
e e $\mathbf{R}^m$ e e *homogeneous vectors* e $\mathbf{R}_1^m$ *inhomogeneous vectors* *vectors*. e e (e) e e e e $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$,
e e $\mathbf{R}^{(m)}$ ($\mathbf{C}^{(m)}$), e e e e e e e
$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, e e e (e) e $\mathbf{R}$ ($\mathbf{C}$). e e e e e $\mathbf{R}^{(m)}$
($\mathbf{C}^{(m)}$), *Clifford number*, e e e, e $x = \sum_S x_S \mathbf{e}_S$, e e $S \neq \emptyset$,
$\mathbf{e}_S$ e *ordered reduced products* e e e e $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, e e S
e e e e e $\{1, 2, \ldots, m\}$, e

$$S = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}, \quad 1 \le l \le m,$$

, $S = \emptyset$, e e $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$.
e e e ee x y $\mathbf{C}^{(m)}$, e e $\langle x, y\rangle$, e
e e $\sum_S x_S \overline{y_S}$, e e $x = \sum_S x_S \mathbf{e}_S$ $y = \sum_S y_S \mathbf{e}_S$. e e
e

$$|x| = \langle x, x\rangle^{1/2} = \left(\sum_S |x_S|^2\right)^{1/2}.$$

e e e $x = x_0 + \underline{x}$, e e e $\overline{x} = x_0 - \underline{x}$. e
e $x \neq 0$ $x \in \mathbf{R}_1^m$, e x e e, x^{-1},

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

e *unit sphere* $\{\underline{x} \in \mathbf{R}^m : |\underline{x}| = 1\}$ e e S^{m-1}. e e $B(\underline{x}, r)$ e e
$\mathbf{R}^m$ e e e $\underline{x}$ r,

e e e e $\mathbf{R}_1^m$, e ee e e e e e $e^{i\langle \underline{x}, \underline{\xi} \rangle}$. e e, $x = x_0\mathbf{e}_0 + \underline{x}$,

$$e(x, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi} \rangle} e^{-x_0|\underline{\xi}|} \chi_+(\underline{\xi}) + e^{i\langle \underline{x}, \underline{\xi} \rangle} e^{x_0|\underline{\xi}|} \chi_-(\underline{\xi}), \tag{2}$$

e e

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm i\frac{\underline{\xi}\mathbf{e}_0}{|\underline{\xi}|}\right).$$

e e e $\chi_\pm$ e e e e

$$\chi_-\chi_+ = \chi_+\chi_- = 0, \quad \chi_\pm^2 = \chi_\pm, \quad \chi_+ + \chi_- = 1.$$

e e $e(\underline{x}, \underline{\xi}) = e^{i\langle \underline{x}, \underline{\xi} \rangle}$ $\mathbf{R}_1^m \times \mathbf{R}^m$, e e , e $\underline{\xi}$, $e(x, \underline{\xi})$ - e - e $x \in \mathbf{R}_1^m$. e e e e e e e - e -1 e e $e(\underline{x}, \underline{\xi})$ $\mathbf{R}_1^m$. e $\mathbf{e}_0$ ϵ_0 e (2), e e ϵ_0 e e e e e e $\mathbf{e}_1, \ldots, \mathbf{e}_m$, $\epsilon_0^2 = -1$ - e e $\mathbf{e}_j$, $j = 1, \ldots, m$, e e e e - e -1 e e $e(x_0\epsilon_0, \underline{x}, \underline{\xi})$ $e^{i\langle \underline{x}, \underline{\xi} \rangle}$ $\mathbf{R}^{m+1}$. $e(x_0\epsilon_0,$

e e, e - e $f(\underline{x}, \underline{y})$ $\tilde{U}$ e $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$, e e $\underline{x} = r\underline{\omega} \in \mathbf{R}^p$, $\underline{y} \in \mathbf{R}^q$, e e $T_k(f)(\underline{\omega}, \underline{y}) = {}_{r\to 0}\, 1/r^k P(k)f(r, \underline{\omega}, \underline{y})$, e e $P(k)$ e e e $M_\ell^+(p, k, \mathbf{C}^{(p)})$. e e e e x. $P_{k,\alpha}(\underline{\omega})$ $\mathcal{M}_\ell^+(p, k, \mathbf{C}^{(p)})$,

$$T_k(f)(\underline{\omega}, \underline{y}) = \sum_{\alpha \in A_k} P_{k,\alpha}(\underline{\omega}) T_{k,\alpha}(f)(\underline{y}),$$

e e $T_{k,\alpha}(f)(\underline{y})$ e e .
e e e e e e e e $T_{k,\alpha}(f)(\underline{y})$ $P_{k,\alpha}(\underline{x})$ $T_{k,\alpha}(\underline{x}, \underline{y})$, e f e e e

$$f(\underline{x}, \underline{y}) = \sum_k \sum_{\alpha \in A_k} T_{k,\alpha}(\underline{x}, \underline{y}), \tag{4}$$

e e $T_{k,\alpha}(\underline{x}, \underline{y}) = \sum_l \underline{x}^l P_{k,\alpha}(\underline{x}) T_{k,\alpha}^{(l)}(f)(\underline{y})$ e e e (4) e e $\tilde{U}$. e e e (4) e *the generalized Taylor series* $T_{k,\alpha}^{(0)}(f)(\underline{y}) = T_{k,\alpha}(f)(\underline{y})$ e e *the initial values* f.
e e e e $\mathbf{R}^p \oplus \mathbf{R}^q$, e ee e e e e e $e^{i\langle \underline{y}, \underline{t}\rangle}$. 4], e k- e e e - e $P_k(\underline{x})$, e e e e e e e e $e^{i\langle \underline{y}, \underline{t}\rangle}$ $\mathcal{T}_{P_k}(\mathbf{R}^q)$, $\underline{x} = r\underline{\omega} \in \mathbf{R}^p$, $\underline{y}, \underline{t} \in \mathbf{R}^q$, e e

$$\varepsilon_{P_k}^p(\underline{x}, \underline{y}, \underline{t}) = \Gamma\left(k + \frac{p}{2}\right) r^k e^{i\langle \underline{y}, \underline{t}\rangle} \left(\frac{r|\underline{t}|}{2}\right)^{-k-(p/2)+1} \left[I_{k+(p/2)-1}(r|\underline{t}|) + i I_{k+(p/2)}(r|\underline{t}|)\underline{\omega}\frac{\underline{t}}{|\underline{t}|}\right] P_k(\underline{\omega}), \tag{ }$$

e e

$$I_v(u) = i^{-v} J_v(iu) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v+k-1)} \left(\frac{u}{2}\right)^{u+2k},$$

e e e . e $\varepsilon_{P_k}^p(\underline{x}, \underline{y}, \underline{t})$ e - e $\mathbf{R}^p \oplus \mathbf{R}^q$.
, e $P_k = 1$, $k = 0$, e e

$$\varepsilon_1^p(\underline{x}, \underline{y}, \underline{t}) = \Gamma\left(\frac{p}{2}\right) e^{i\langle \underline{y}, \underline{t}\rangle} \left(\frac{r|\underline{t}|}{2}\right)^{-(p/2)+1} \left[I_{(p/2)-1}(r|\underline{t}|) + i I_{p/2}(r|\underline{t}|)\underline{\omega}\frac{\underline{t}}{|\underline{t}|}\right].$$

e e e

$$\varepsilon_1^1(x_1\mathbf{e}_1, \underline{y}, \underline{t}) = e(x_1\mathbf{e}_1, \underline{y}, \underline{t}).$$

4], e e, $u > 0$,

$$\left(\frac{u}{2}\right)^{-v} I_v(u) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v+k+1)} \left(\frac{u}{2}\right)^{2k} \le C \sum_{k=0}^{\infty} \frac{u^{2k}}{(2k)!} \le C e^u. \tag{ }$$

e $|\underline{t}| \leq \Omega$, e e

$$\begin{aligned}|\varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{t})| &\leq C\left(\frac{2}{\Omega}\right)^{k}\left(\frac{r\Omega}{2}\right)^{-(p/2)+1}\left[I_{k+\frac{p}{2}-1}(r\Omega) + I_{k+\frac{p}{2}}(r\Omega)\right]\\ &\leq C[I_k(r\Omega) + I_{k+1}(r\Omega)]\\ &\leq Ce^{r\Omega}. \qquad ( )\end{aligned}$$

3. Exact interpolation with Shannon sampling in $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$

e e e e e e e e ( ), e *generalized co-dimension-p sinc function* e $P_k(\underline{x}) \in M^{+}_{\ell}(p, k, \mathbf{C}^{(p)})$ e e

$$^{p}_{P_k}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{t})\chi_{[-\pi,\pi]^q}(\underline{t}) \ \underline{t}. \qquad ( )$$

h 0 e , e e e *cardinal function* f e

$$C(f, h)(\underline{x}, \underline{y}) \equiv \sum_{\underline{k}\in\mathbf{Z}^q} {}^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) f(h\underline{k}),$$

e ( ), e e

$$^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) = \frac{h^q}{(2\pi)^q}\int_{[-(\pi/h,(\pi/h]^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y} - h\underline{k}, \underline{t}) \ \underline{t}. \qquad ( )$$

e , e e e e e e - e -*p* e e e e e e e - e -*p* e e e 4]

1 4] (e e e - e -*p* e e) *Let* $P_k \in M^{+}_{\ell}(p, k\ \mathbf{C}^{(p)})$ *be given, F analytic, defined in* $\mathbf{R}^q$, *taking values in* $\mathbf{C}^{(q)}$, *which is the complex Clifford algebra generated by* $\mathbf{e_{p+1}}, \ldots, \mathbf{e_{p+q}}$, *and* $F \in L^2(\mathbf{R}^q)$, Ω *be a positive real number. Then the following two assertions are equivalent*

1^0 *F has a homogeneous co-dimensional-p generalized CK extension to* $\mathbf{R}^{p+q}$, *denoted by* f_{P_k}, *and there exists a constant C such that*

$$|f_{P_k}(\underline{x}, \underline{y})| \leq Ce^{\Omega|\underline{x}|}, \text{ for any } \underline{x} \in \mathbf{R}^p,\ \underline{y} \in \mathbf{R}^q.$$

2^0 *supp*$(\hat{F}) \subset \overline{B}(0, \Omega)$.

Moreover, if one of the above conditions holds, then we have

$$f_{P_k}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{\xi})\hat{F}(\underline{\xi}) \ \underline{\xi}, \text{ for any } \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q.$$

f $\mathbf{R}^p \oplus \mathbf{R}^q$ e *exponential type* Ω

$$|f(\underline{x}, \underline{y})| \leq Ce^{\Omega|\underline{x}|}, \qquad \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$$

.

h 0, e e

$$PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h) = \{f \mid f \in \mathcal{T}_{P_k}(\mathbf{R}^q) \quad \text{e} \quad \text{e} \quad \text{e}\ \pi/h, \quad \text{e} \quad \text{e} \quad F \in L^2(\mathbf{R}^q) \quad \text{e} \quad \mathbf{C}^{(q)}\}.$$

, $k = 0,\ P_k = 1$ $p = 1$, e e

$$PW_{\mathbf{R}^{q+1}}(\pi/h) = \{f \mid f \quad \text{e -} \quad \text{e} \quad \mathbf{R}^{q+1} \quad \text{e} \quad \text{e} \quad \text{e}\ \pi/h,\ f|_{\mathbf{R}^q} \in L^2(\mathbf{R}^q),\ f|_{\mathbf{R}^q} \quad \text{e} \quad \mathbf{C}^{(q)}\}.$$

e e e e e e e $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$.

1 *If* $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, *then for any* $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$, *we have*

1^0

$$f(\underline{x}, \underline{y}) = \frac{1}{h^q} \int_{\mathbf{R}^q} \quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - \underline{\xi}}{h}\right) F(\underline{\xi}) \quad \underline{\xi}.$$

2^0

$$\frac{1}{(2\pi)^q} \int_{[-\pi/h,\, \pi/h]^q} |\hat{F}(\underline{y})|^2 \quad \underline{y} = \int_{\mathbf{R}^q} |F(\underline{t})|^2 \quad \underline{t} = \sum_{\underline{k} \in \mathbf{Z}^q} |F(h\underline{k})|^2, \tag{10}$$

where F is the initial value of f.

Proof 1^0 e $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, e 1, e e

$$f(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q} \int_{\overline{B}(0,\, \pi/h)} \varepsilon^{p}_{P_k}(\underline{x}, \underline{y}, \underline{t}) \hat{F}(\underline{t}) \quad \underline{t}$$

$$= \frac{1}{(2\pi)^q} \int$$

2^0 e 1, e e

$$F(\underline{t}) = \frac{1}{(2\pi)^q}\int_{B(0,\,\pi/h)} e^{i\langle \underline{y},\,\underline{t}\rangle}\hat{F}(\underline{y})\ \underline{y} = \frac{1}{(2\pi)^q}\int_{[-\pi/h,\,\pi/h]^q} e^{i\langle \underline{y},\,\underline{t}\rangle}\hat{F}(\underline{y})\ \underline{y}.$$

e e e e $\hat{F}$ e e $[-\pi/h, \pi/h]^q$, e e

$$h^q F(h\underline{k}) = \frac{1}{(2R)^q}\int_{[-R,\,R]^q} e^{i\pi\langle \underline{x},\,\underline{k}\rangle/R}\hat{F}(\underline{y})\ \underline{y} = c_k,$$

e e $R = \frac{\pi}{h}$, c_k e e e e e $\hat{F}$. e e e e e
e e e

$$\int_{[-R,\,R]^q} |\hat{F}(\underline{y})|^2\ \underline{y} = (2R)^q \sum_{\underline{k}\in\mathbf{Z}^q} |c_k|^2,$$

e e e e e L^2- $\mathbf{R}^q$ e

$$\int_{\mathbf{R}^q} |\hat{F}(\underline{y})|^2\ \underline{y} = \int_{[-R,\,R]^q} |\hat{F}(\underline{y})|^2\ \underline{y} = (2\pi)^q \int_{\mathbf{R}^q} |F(\underline{t})|^2\ \underline{t}.$$

e e

$$\frac{1}{(2\pi)^q}\int_{[-\pi/h,\,\pi/h]^q} |\hat{F}(\underline{y})|^2\ \underline{y} = \int_{\mathbf{R}^q} |F(\underline{t})|^2\ \underline{t} = \sum_{\underline{k}\in\mathbf{Z}^q} |F(h\underline{k})|^2.$$

1 *If $f \in PW_{\mathcal{T}_1(\mathbf{R}^q)}(\pi/h)$, then for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$, we have*

$$f(\underline{x}, \underline{y}) = \frac{1}{h^q}\int_{\mathbf{R}^q} {}^{p}_{1}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - \underline{\xi}}{h}\right) f(\underline{\xi})\ \underline{\xi}.$$

$$\frac{1}{(2\pi)^q}\int_{[-\pi/h,\,\pi/h]^q} |\hat{f}(\underline{y})|^2\ \underline{y} = \int_{\mathbf{R}^q} |f(\underline{t})|^2\ \underline{t} = \sum_{\underline{k}\in\mathbf{Z}^q} |f(h\underline{k})|^2.$$

( ) e - e -p e e , e ${}^{p}_{P_k}(\underline{x}/h, \underline{y}/h)$
e $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$. e e, e
$PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$ e e e .

2 *Let $P_k \in M_\ell^+(p, k\ \mathbf{C}^{(p)})$ be given, $F \in L^2(\mathbf{R}^q)$ and take values in $\mathbf{C}^{(q)}$. Then $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$, where*

$$f(\underline{x}, \underline{y}) = h^q \int_{\mathbf{R}^q} {}^{p}_{P_k}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - \underline{\xi}}{h}\right) F(\underline{\xi})\ \underline{\xi}. \tag{11}$$

Proof e e ’ e e e - e e (11),
e ( ), e e

$$
\begin{aligned}
f(\underline{x},\underline{y}) &= \frac{h^q}{(2\pi)^q}\int_{\mathbf{R}^q}\widehat{\left[\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-\underline{\xi}}{h}\right)\right]}(-\underline{t})\hat{F}(\underline{t})\ \ \underline{t}\\
&= \frac{h^q}{(2\pi)^q}\int_{\mathbf{R}^q} h^{-q}\varepsilon^{p}_{P_k}(\underline{x},\underline{y},\underline{t})\chi_{[-\pi/h,\pi/h]^q}(\underline{t})\hat{F}(\underline{t})\ \ \underline{t}\\
&= \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q} \varepsilon^{p}_{P_k}(\underline{x},\underline{y},\underline{t})\chi_{[-\pi/h,\pi/h]^q}(\underline{t})\hat{F}(\underline{t})\ \ \underline{t}\\
&= \frac{1}{(2\pi)^q}\int_{[-\pi/h,\pi/h]^q} \varepsilon^{p}_{P_k}(\underline{x},\underline{y},\underline{t})\hat{F}(\underline{t})\ \ \underline{t}.
\end{aligned}
$$

e e ( ) $\varepsilon^{p}_{P_k}$, e e

$$
|f(\underline{x},\underline{y})| \le Ce^{\sqrt{q}\pi/h|\underline{x}|}, \qquad \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.
$$

e 1, e e $f(\underline{x},\underline{y}) \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\sqrt{q}\pi/h)$. ■

e , e e $_{P_k}$ e $PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$ e .

3 *If $f \in PW_{\mathcal{T}_{P_k}(\mathbf{R}^q)}(\pi/h)$, then for any $\underline{x}\in\mathbf{R}^p$, $\underline{y}\in\mathbf{R}^q$,*

$f(\underline{x},\underline{y}) = C(f,h)(\underline{x},\underline{y}$ [redacted] $\sum_{\underline{k}\in\mathbf{Z}^q} {}^{p}_{P_k}\ \frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}$ fXð

e e e e (14) e e (13) e e e
e e e e e L^2- e e e, e e

$$f(\underline{x},\underline{y}) = \frac{1}{(2\pi)^q}\int_{[-\pi/h,\,\pi/h]^q}\sum_{\underline{k}\in\mathbf{Z}^q} e^{i\langle h\underline{k},\,\underline{t}\rangle}\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\hat{F}(\underline{t})\ \ \underline{t}$$

$$= \sum_{\underline{k}\in\mathbf{Z}^q}\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\left(\frac{1}{(2\pi)^q}\int_{[-\pi/h,\,\pi/h]^q} e^{i\langle h\underline{k},\,\underline{t}\rangle}\hat{F}(\underline{t})\ \ \underline{t}\right)$$

$$= \sum_{\underline{k}\in\mathbf{Z}^q}\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)F(h\underline{k}).$$

e e e e e e e e e e e.
, e e M, e e , e e

$$\left|\sum_{|\underline{k}|\rangle M}\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)F(h\underline{k})\right| \le \left(\sum_{|\underline{k}|\rangle M}\left|\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\right|^2\right)^{1/2}\left(\sum_{|\underline{k}|\rangle M}|F(h\underline{k})|^2\right)^{1/2}.$$

e e $\varepsilon^p_{P_k}(\underline{x},\underline{y},\cdot)\in L^2([-\pi/h,\pi/h]^q)$. e e e e
e ( ), e e $U\in\mathbf{R}^p$, e e

$$\left(\sum_{|\underline{k}|\rangle M}\left|\quad {}^{p}_{P_k}\left(\frac{\underline{x}}{h},\frac{\underline{y}-h\underline{k}}{h}\right)\right|^2\right)^{1/2} \le \left(\frac{h}{2\pi}\right)^{q/2}\|\varepsilon^p_{P_k}(\underline{x},\underline{y},\cdot)\|_{L^2([-\pi/h,\,\pi/h]^q)}$$

$$\le \left(\frac{h}{2\pi}\right)^{q/2} e^{\sqrt{q}|\underline{x}|\pi/h} \le C < \infty,$$

e e $\underline{y}\in\mathbf{R}^q$, $\underline{x}\in U$. e e e e (10), e e e e (12)
e e e $U\oplus\mathbf{R}^q$. ■

e e e - e -*p* e e e

2 4] (e e - e -*p* e e) *Let F be analytic, defined in* $\mathbf{R}^q$, *taking values in* $\mathbf{C}^{(q)}$, *the complex Clifford algebra generated by* $\mathbf{e_{p+1}},\ldots,\mathbf{e_{p+q}}$, *and* $F\in L^2(\mathbf{R}^q)$. Ω *is a positive real number. Then the following two assertions are equivalent*

1^0 *F has a homogeneous co-dimensional-p CK extension to* $\mathbf{R}^{p+q}$, *denoted by f, and there exists a constant C such that*

$$|f(\underline{x},\underline{y})| \le Ce^{\Omega|\underline{x}|},\ \textit{for any}\ \underline{x}\in\mathbf{R}^p,\ \underline{y}\in\mathbf{R}^q.$$

2^0 $supp(\hat{F})\subset\overline{B}(0,\Omega)$.

Moreover, if one of the above conditions holds, we have

$$f(\underline{x},\underline{y}) = \frac{1}{(2\pi)^q}\int_{\mathbf{R}^q}\varepsilon^p_1(\underline{x},\underline{y},\underline{\xi})\hat{F}(\underline{\xi})\ \ \underline{\xi},\ \textit{for any}\ \underline{x}\in\mathbf{R}^p, \underline{y}\in\mathbf{R}^q.$$

e e e 2, e e e e $_1^p$ e
$PW_{T_1(\mathbf{R}^q)}(\pi/h)$.

2 *If* $f \in PW_{T_1(\mathbf{R}^q)}(\pi/h)$, *then for any* $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$,

$$f(\underline{x}, \underline{y}) = C(f, h)(\underline{x}, \underline{y}) = \sum_{\underline{k} \in \mathbf{Z}^q} \ {}_1^p\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) f(h\underline{k}),$$

where

$$_1^p(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q} \int_{\mathbf{R}^q} \varepsilon_1^p(\underline{x}, \underline{y}, \underline{t}) \chi_{[-\pi,\pi]^q}(\underline{t}) \ \underline{t},$$

and the series on the right-hand side is absolutely and uniformly convergent for any $\underline{y} \in \mathbf{R}^q$ *and* $\underline{x}$ *belongs to any bounded set in* $\mathbf{R}^p$.

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3 *Assume that* $f(\underline{x}, \underline{y})$ *is left-monogenic in* $\mathbf{R}^m = \mathbf{R}^p \oplus \mathbf{R}^q$ *with the form* (4). *For any* $k \geq 0$ *and* $\alpha \in A_k$, *let* $T_{k,\alpha}(f)(\underline{y}) = T_{k,\alpha}^{(0)}(f)(\underline{y})$ *be analytic, defined in* $\mathbf{R}^q$, *taking values in* $\mathbf{C}^{(q)}$, *the complex Clifford algebra generated by* $\mathbf{e_{p+1}}, \ldots, \mathbf{e_{p+q}}$, $T_{k,\alpha}(f)(\underline{y}) \in L^2(\mathbf{R}^q)$,

$$\left| \sum_k \sum_\alpha P_{k,\alpha}(\underline{x}) \hat{T}_{k,\alpha}(f)(\underline{\xi}) \right| \leq Ce^{\Omega|\underline{x}|}, \ \textit{for any } \underline{x} \in \mathbf{R}^p, \ \underline{\xi} \in \mathbf{R}^q,$$

where Ω *is a positive real number. Then the following two assertions are equivalent*

1^0 *There exists a constant C such that*

$$|f(\underline{x}, \underline{y})| \leq Ce^{\Omega|\underline{x}|}, \ \textit{for any } \underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q.$$

2^0 $(\hat{T}_{k,\alpha}(f)) \subset \overline{B}(0, \Omega)$, *for any* $k \geq 0$ *and* $\alpha \in A_k$.

Moreover, if one of the above conditions holds, we have

$$f(\underline{x}, \underline{y}) = \sum_k \sum_\alpha T_{k,\alpha}(\underline{x}, \underline{y}) = \frac{1}{(2\pi)^q} \sum_k \sum_\alpha \int_{\mathbf{R}^q} \varepsilon_{P_{k,\alpha}}^p(\underline{x}, \underline{y}, \underline{\xi}) \hat{T}_{k,\alpha}(f)(\underline{\xi}) \ \underline{\xi}, \qquad (1\ )$$

for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$ *and the series is converging uniformly on any compact set in* $\mathbf{R}^p \oplus \mathbf{R}^q$.

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$PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h) = \{f|f$ e - e $\mathbf{R}^p \oplus \mathbf{R}^q$ e (4) e e e π/h, e e $T_{k,\alpha}(f)(\underline{y}) \in L^2(\mathbf{R}^q)$ e $\mathbf{C}^{(q)}$.}

4 *If $f \in PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h)$, then for any $\underline{x} \in \mathbf{R}^p, \underline{y} \in \mathbf{R}^q$,*

$$f(\underline{x}, \underline{y}) = \sum_k \sum_\alpha C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}), \tag{1 }$$

where

$$C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^q} \quad {}^{p}_{P_{k,\alpha}}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) T_{k,\alpha}(f)(h\underline{k}) \tag{1 }$$

and $T_{k,\alpha}(f)(\underline{y})$ are the initial values of f. The series (1) *and* (1) *on the right-hand side are uniformly convergent on any compact set in $\mathbf{R}^p \oplus \mathbf{R}^q$.*

Proof $f \in PW_{\mathbf{R}^p\oplus\mathbf{R}^q}(\pi/h)$, e f e (1). e e 3, e

$$T_{k,\alpha}(\underline{x}, \underline{y}) = C\big[T_{k,\alpha}(f), h\big](\underline{x}, \underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^q} \quad {}^{p}_{P_{k,\alpha}}\left(\frac{\underline{x}}{h}, \frac{\underline{y} - h\underline{k}}{h}\right) T_{k,\alpha}(f)(h\underline{k}).$$

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