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Direct Sum Decomposition of $L^2(\mathbf{R}_1^n)$ into Subspaces Invariant under Fourier Transformation

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ABSTRACT. *Denote by* $\mathbf{R}_1^n$ *the real-linear span of* $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$, *where* $\mathbf{e}_0 = 1, \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}, 1 \leq i, j \leq n$. *Under the concept of left-monogeneity defined through the generalized Cauchy-Riemann operator we obtain the direct sum decomposition of* $L^2(\mathbf{R}_1^n)$, $n > 1$,

$$L^2(\mathbf{R}_1^n) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k ,$$

where Ω^k *is the right-Clifford module of finite linear combinations of functions of the form* $R(x)h(|x|)$, *where, for* $d = n+1$, *the function* R *is a* k- *or* $-(d+|k|-2)$-*homogeneous left-monogenic function, for* $k > 0$ *or* $k < 0$, *respectively, and* h *is a function defined in* $[0, \infty)$ *satisfying a certain integrability condition in relation to* k, *the spaces* Ω^k *are invariant under Fourier transformation. This extends the classical result for* $n = 1$. *We also deduce explicit Fourier transform formulas for functions of the form* $R(x)h(r)$ *refining Bochner's formula for spherical* k-*harmonics.*

1. Introduction

Fourier analysis in Euclidean spaces is intimately connected with the action of the group the special features of the theory is the invariance of Fourier transformation on certain subspaces of the square integrable functions, defined through radial functions, spherical harmonics and Bessel functions. The latter is regarded as the symmetric property of Fourier transformation [8, 3].

In the one-dimensional Euclidean space a function may be decomposed into a sum of an even function and an odd function. It is easy to verify that the Fourier transform of an even function is still an even function, and that of an odd function is still an odd function (We suppress the words “square integrable” in front of “function” when we discuss Fourier interests in practice, as functions are decomposed into components of different phases. In

(i) M_k^+ the space of left-monogenic homogeneous polynomials in $\mathbf{R}_1^n$ of degree k. An arbitrary element of it, denoted by P_k, is called a left-inner monogenics of degree k.
(ii) M_k^- the space of left-monogenic homogeneous functions in $\mathbf{R}_1^n \setminus \{0\}$ of degree $-(d+k-1)$. An arbitrary element of it, denoted by Q_k, is called a *left-outer monogenics of degree k*.
(iii) $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ the spaces consisting of the restrictions to the unit sphere Σ_n of, respectively, the functions in M_k^+ and M_k^-. The elements of $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ are called *spherical monogenics,* or *surface spherical monogenics.*
(iv) $\mathcal{H}_k$ the space of surface spherical harmonics of degree k in $\mathbf{R}_1^n$.

For the lowest dimension $n = 1$ we recall the following result [8].
For $k \in \mathbf{Z}$, let

$$\Omega^k = \left\{ g \in L^2(\mathbf{R}_1^1) : g(z) = f(r)e^{ik\theta} \text{ for some measurable function } f(r) \right.$$
$$\left. \text{satisfying} \int_0^\infty |f(r)|^2 r\,dr < \infty \right\}.$$

We have the following.

Proposition 1. *The direct sum decomposition*

$$L^2(\boldsymbol{R}_1^1) = \sum_{k=-\infty}^{\infty} \bigoplus \Omega^k \tag{2.1}$$

holds in the sense that:
(a) *The subspaces Ω^k are closed.*
(b) *The subspaces Ω^k are mutually orthogonal, $k \in \mathbf{Z}$.*
(c) *Every function of $L^2(\boldsymbol{R}_1^1)$ is a limit of finite linear combinations of functions in $\bigcup_{k=-\infty}^{\infty} \Omega^k$.*
(d) *Fourier transformation maps each subspace Ω^k into itself.*

Furthermore, we have the following.

Proposition 2. *For any $f \in \Omega^k$ of the form $f(z) = f_0(r)e^{ik\theta}$, where $z = re^{i\theta}$, then $\hat{f}(\omega) = F_0(R)e^{ik\phi}$, where $\omega = \mathrm{Re}^{i\phi}$,*

$$F_0(R) = 2\pi i^{-k} \int_0^\infty f_0(r) J_k(2\pi Rr) r\,dr\ ,$$

where $J_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta} e^{-ik\theta}\,d\theta$ is the Bessel function of order k, $k \in \mathbf{Z}$.

For the spaces $\mathbf{R}_1^n$, $n > 1$, the result is not quite the same. There holds [8]

$$L^2(\mathbf{R}_1^n) = \sum_{k=0}^{\infty} \bigoplus \mathcal{N}_k\ , \tag{2.2}$$

where $\mathcal{N}_k$, $k \geq 0$, is the right-Clifford module of finite linear combinations of functions of the form $H(x)f(r)$, where f is a function, defined on $[0, \infty)$, satisfying $\int_0^\infty |f(r)|^2 r^{d+2k-1}\,dr < \infty$ and H a *solid harmonics of degree k.* Moreover, for $f \in \mathcal{N}_k$

of the form $f(x) = H_k(x) f_0(r)$, where $x = rx'$ and H_k is a k-harmonics, there holds Bochner's formula in terms of spherical harmonics

$$\hat{f}(x) = H_k(x) g_0(r) ,$$

where

$$g_0(r) = 2\pi i^{-k} r^{-[(d+2k-2)/2]} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{(d+2k)/2} \, ds ,$$

and

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its} \left(1 - s^2\right)^{(2k-1)/2} ds$$

is a form of Bessel function of order k, $k \in \mathbf{N}$.

(ii) [illegible] 0, to be the right Clifford module of finite linear combinations of functions of

Since $Y \in \mathcal{M}_{k-1}^-$, by (2.3), we have $Y \in \mathcal{H}_k$. Recalling the proof of (2.2) (Lemma 2.18, Chapter IV, [8]), there exists a function φ, defined on $[0, \infty)$, such that

$$\begin{aligned} \hat{f}(x) &= \int_0^\infty \left\{ \int_{\Sigma_n} e^{-2\pi r\rho x'\cdot u'} Y(u')\, du' \right\} f_0(\rho)\rho^{-(k-1)}\, d\rho \\ &= \int_0^\infty \left(\varphi(r\rho)Y(x')\right) f_0(\rho)\rho^{-(k-1)}\, d\rho \end{aligned}$$

$$= Q(x)\left\{ r^{d+k-2} \int_0^\infty f_0(\rho)\rho^{-(k-1)}\varphi(r\rho)\, d\rho \right\}.$$

Let $g_0(r) = r^{d+k-2}\int_0^\infty f_0(\rho)\rho^{-(k-1)}\varphi(r\rho)\,d\rho$. Since $f \in L^2(\mathbf{R}_1^n)$, by the Plancherel Theorem, $\|\hat{f}\|_2 < \infty$, where $\hat{f} = Q(x)g_0(r)$ and $Q(x) \in M_{k-1}^-$. With $\|Q\|_\Sigma = (\int_{\Sigma_n} |Q(x')|^2\, dx')^{1/2}$ and $\|Q\|_\Sigma < \infty$, we have

$$\begin{aligned} \|\hat{f}\|_2^2 &= \int_{\mathbf{R}_1^n} |g(r)Q(x)|^2\, dx \\ &= \left(\int_0^\infty |g(r)r^{-(d+2k-3)}|\, dr \right) \|Q\|_\Sigma^2 < \infty\,. \end{aligned}$$

So, $\int_0^\infty |g(r)|^2 r^{-(d+2k-3)}\, dr < \infty$. This shows $\hat{f} \in \Omega^{-k}, k \in \mathbf{N}$. □

When $k \geq 0$, the space Ω^k, being isomorphic to M_k^+, has the dimension $\alpha_k = C_{d+k-2}^k$ [2]. Let $\{P^{(1)}, P^{(2)}, \ldots, P^{(\alpha_k)}\}$ be an orthonormal basis of the space. A general function $F(x)$ in Ω^k can be uniquely written in the form $\sum_{j=0}^{\alpha_k} P^{(j)}(x) f_j(r)$, and

$$\|F\|_2^2 = \int_{\mathbf{R}_1^n} |F(x)|^2\, dx = \sum_{j=1}^{\alpha_k} \int_0^\infty |f_j(r)|^2 r^{d+2k-1}\, dr\,. \tag{3.1}$$

Similarly, when $k \in \mathbf{N}$, the space Ω^{-k}, being isomorphic to M_{k-1}^-, has the dimension $\beta_k = C_{d+k-3}^{k-1}$ [2]. Let $\{Q^{(1)}, Q^{(2)}, \ldots, Q^{(\beta_k)}\}$ be an orthonormal basis of the space. Any typical function $G(x)$ in Ω^{-k} can be uniquely written in the form $\sum^{\beta_k} Q^{(j)}(x) g_j(r)$, and

Using (3.1) and (3.2), we further write, for $k \geq 0$,

$\|f^{(k)}\|_2^2 = \sum_{j=1}^{\alpha_k} \int_0^\infty |f_j^{(k)}(r)|^2 r^{d+2k-1}\,dr$; and, for $k < 0$, $\|f^{(k)}\|_2^2 = \sum_{j=1}^{\beta_{|k|}} \int_0^\infty |g_j^{(|k|)}(r)|^2 r^{-(d+2|k|-3)}\,dr$.

In the proof of Theorems 1 and 2 we know little about the function φ, so we did not get the explicit representation of $g_0(r)$ in terms of f_0. In below we will concentrate in obtaining such explicit formulae. When $k \geq 0$, $\Omega^k \subset \mathcal{N}_k$, and hence any function f in Ω^k is also in $\mathcal{N}_k$. Thus, Bochner's formula on harmonic polynomials can be used: For$f(x) = H_k(x) f_0(r)$, where H_k is either a k-harmonics or k-monogenics, we have $\hat{f}(x) = H_k(x) g_0(r)$, where

$$g_0(r) = 2\pi i^{-k} r^{-[(d+2k-2)/2]} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{(d+2k)/2}\,ds\,. \tag{3.3}$$

Next we consider the case Ω^{-k}, $k \in \mathbf{N}$. We need more information on bases of Ω^{-k}.

Definition 3. Let

$$\omega_0(x) = E(x)$$
$$\omega_{l_1 \ldots l_k}(x) = (-1)^k \frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} E(x)\,,$$

where $(l_1, \ldots, l_k) \in \{1, 2, \ldots, n\}^k$, $k \in \mathbf{N}$.

It is deduced in [2] that $\{\omega_{l_1 \ldots l_k} : (l_1, \ldots, l_k) \in \{1, 2, \ldots, n\}^k\}$ is a basis of M_k^-. For $x \in \mathbf{R}_1^n \setminus \{0\}$,

$$\begin{aligned}
\omega_{l_1 \ldots l_k}(x) &= \frac{(-1)^k}{A_n} \frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} \frac{\bar{x}}{|x|^{n+1}} \\
&= \frac{1}{n-1} \frac{(-1)^{k+1}}{A_n} \bar{D} \left[\frac{\partial}{\partial x_{l_1}} \cdots \frac{\partial}{\partial x_{l_k}} \frac{1}{|x|^{n-1}} \right] \\
&= \bar{D} \left[\frac{H_k^{(l_1 \ldots l_k)}(x)}{|x|^{n+2k-1}} \right] \\
&= \frac{1}{|x|^{n+2k+1}} \left[|x|^2 \bar{D} H_k^{(l_1 \ldots l_k)}(x) - (n+2k-1) \bar{x} H_k^{l_1 \ldots l_k}(x) \right] \\
&= \frac{H_{k+1}^{(l_1 \ldots l_k)}(x)}{|x|^{n+2k+1}}\,,
\end{aligned}$$

where $H_k^{(l_1 \ldots l_k)}(x)$ and $H_{k+1}^{(l_1 \ldots l_k)}(x)$ are polynomials of homogeneity k and $k+1$, respectively. We will show that both of them are harmonic.

Lemma 1. *Let $G(x) = \frac{P(x)}{|x|^{n+2k-1}}$, $x \in \mathbf{R}_1^n \setminus \{0\}$, where $P(x)$ is a homogeneous polynomial of degree k, then*

$$\Delta G(x) = \frac{\Delta P(x)}{|x|^{n+2k-1}}\,,$$

where Δ is the Laplacian for $n+1$ variables $x_0, x_1, \ldots, x_n$.

Proof. Consecutively taking partial derivatives, we have, for any $i = 0, 1, \ldots, n$,

$$\frac{\partial}{\partial x_i}G(x) = \left(\frac{\partial}{\partial x_i}P(x)\right)\frac{1}{|x|^{n+2k-1}} + P(x)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}},$$

$$\frac{\partial^2}{\partial x_i^2}G(x) = \left(\frac{\partial^2}{\partial x_i^2}P(x)\right)\frac{1}{|x|^{n+2k-1}} + \left(\frac{\partial}{\partial x_i}P(x)\right)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}}$$
$$+\left(\frac{\partial}{\partial x_i}P(x)\right)[-(n+2k-1)]\frac{x_i}{|x|^{n+2k+1}}$$
$$+P(x)[-(n+2k-1)]\left[\frac{1}{|x|^{n+2k+1}} - (n+2k+1)\frac{x_i^i}{|x|^{n+2k+3}}\right]$$
$$= \left(\frac{\partial^2}{\partial x_i^2}P(x)\right)\frac{1}{|x|^{n+2k-1}} - 2(n+2k-1)\frac{1}{|x|^{n+2k+1}}x_i\frac{\partial}{\partial x_i}P(x)$$
$$-(n+2k-1)\frac{P(x)}{|x|^{n+2k+1}} + \frac{(n+2k-1)(n+2k+1)}{|x|^{n+2k+3}}x_i^2P(x).$$

Then

$$\Delta G(x) = [\Delta P(x)]\frac{1}{|x|^{n+2k-1}} - 2(n+2k-1)\frac{1}{|x|^{n+2k+1}}\left[x_0\frac{\partial}{\partial x_0}P(x) + \cdots + x_n\frac{\partial}{\partial x_n}P(x))\right]$$
$$-(n+1)(n+2k-1)\frac{P(x)}{|x|^{n+2k+1}} + \frac{(n+2k-1)(n+2k+1)}{|x|^{n+2k+3}}(x_0^2+\cdots+x_n^2)P(x)$$
$$= \frac{\Delta P(x)}{|x|^{n+2k-1}} - \frac{2(n+2k-1)}{|x|^{n+2k+1}}\left[\left(x_0\frac{\partial}{\partial x_0}P(x)+\cdots+x_n\frac{\partial}{\partial x_n}P(x)\right) - kP(x)\right].$$

Since $P(x)$ is homogeneous of degree k, by Euler's formula, we have that

$$\sum_{i=0}^{n} x_i\frac{\partial P(x)}{\partial x_i} = kP(x),$$

i.e.,
$$\left(x_0\frac{\partial}{\partial x_0}P(x)+\cdots+x_n\frac{\partial}{\partial x_n}P(x)\right) - kP(x) = 0.$$

Therefore, we get that

$$\Delta G(x) = \frac{\Delta P(x)}{|x|^{n+2k-1}}.$$

□

Corollary 1. *Functions $H_k^{(l_1\ldots l_k)}(x)$ and $H_{k+1}^{(l_1\ldots l_k)}(x)$ are harmonic.*

Proof. Set

$$g_{l_1\ldots l_k}(x) = \frac{H_k^{(l_1\ldots l_k)}(x)}{|x|^{n+2k-1}}.$$

Since $\omega_{l_1\ldots l_k}$ is left-monogenic in $\mathbf{R}_1^n \setminus \{0\}$, it follows from $\Delta = D\bar{D}$ that

$$\omega = \bar{D}g \quad \text{and} \quad \Delta g_{l_1\ldots l_k}(x) = D\omega_{l_1\ldots l_k}(x) = 0,\ x \in \mathbf{R}_1^n \setminus \{0\}.$$

Therefore, $g_{l_1\ldots l_k}$ is harmonic in $\mathbf{R}_1^n \setminus \{0\}$. From Lemma 1 we conclude that $H_k^{(l_1\ldots l_k)}(x)$ is harmonic. Since $\omega_{l_1\ldots l_k}$ is harmonic, the lemma implies that $H_{k+1}^{(l_1\ldots l_k)}(x)$ is harmonic. □

The corollary can also be established by noting that if $P(x)$ is left monogenic then $xP(x)$ is harmonic and then applying Kelvin inversion. That $xP(x)$ is harmonic was first deduced by A. Sudbery in [7] for quaternionic case and was extended to higher-dimensional cases by J. Ryan in [5].

The following extends the classical Bochner's formula to homogeneous monogenic functions of negative degrees.

Theorem 3. *Let $f \in \Omega^{-k}$ of the form $f(x) = Q(x)f_0(|x|)$, $Q(x) \in M^-_{k-1}$. Then*

$$\hat{f}(x) = Q(x)g_0(|x|)\,,$$

where, with $r = |x|$,

$$g_0(r) = 2\pi i^{-k} r^{(d+2k-2)/2} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{-[(d+2k-4)/2]}\, ds\,, \tag{3.4}$$

where

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}\left(1-s^2\right)^{(2k-1)/2} ds$$

is the Bessel function of order k.

Proof. From [2], the $-(d+k-2)$-homogeneous functions $\omega_{l_1 \ldots l_k}$, $(l_1, \ldots, l_{k-1}) \in \{1, 2, \ldots, n\}^{k-1}$, form a basis of M^-_{k-1}. As a function in M^-_{k-1}, the function Q has the form

$$Q(x) = \frac{H_k(x)}{|x|^{d+2k-2}}\,,$$

[illegible] Invoking the second assertion

[illegible]

[illegible]

The formulas (3.3) and (3.4) together provide a refinement of Bochner's formula with spherical harmonics replaced by spherical monogenics. The formulas (3.3) and (3.4) can be unified into one formula by using the signum function.

Let $f \in \Omega^k, k \in \mathbf{Z}$, and $f(x) = R(x)h(r)$, where if $k \geq 0$, then $R(x) \in M_k^+$; and, if $k < 0$, then $R(x) \in M_{|k|-1}^-$. Then we have

$$\hat{f}(x) = R(x)H(r) \, ,$$

and, with $c_k = (d + 2|k| - 2)/2, k \in \mathbf{Z}$,

$$H(r) = 2\pi i^{-k} r^{-\operatorname{sgn}(k)c_k} \int_0^\infty f_0(s) J_{c_k}(2\pi r s) s^{1+\operatorname{sgn}(k)c_k} \, ds \, ,$$

where $\operatorname{sgn}(k)$ is the signum function that takes the value $+1$, -1 or 0 for $k > 0$, $k < 0$ or $k = 0$.

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