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Analytic signals and harmonic measures ☆

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Abstract

We prove that a sufficient and necessary condition for $He^{\;(\;)} = -\;e^{\;(\;)}$, where H is Hilbert transformation, is a continuous and strictly increasing function with $|\;(\mathbf{R})| = 2$, is that $d\;(\;)$ is a harmonic measure on the line. The counterpart result for the periodic case is also established. The study is motivated by, and has significant impact to time–frequency analysis, especially to aspects of analytic signals inducing instantaneous amplitude and frequency. As a by-product we introduce the theory of Hardy-space-preserving weighted trigonometric series and Fourier transformations induced by harmonic measures in the respective contexts.

Keywords: Harmonic measure; Möbius transformation; Trigonometric system; Hilbert transformation; Time–frequency analysis

1. Introduction

In time–frequency analysis the concept of analytic signals is introduced (see [6]). For a square integrable signal f in the whole time range, the function $f + \; Hf$, where H is Hilbert transformation on the line, is the boundary value of an analytic function in the upper-half complex plane. This may be easily verified from Cauchy's integral on the upper-half complex plane with the boundary data f. Define $A(f) = f + \; Hf$ to be the *analytic*

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signal associated with f. For periodic functions there is an analogous theory. On the space of square integrable signals on $[0, 2\ \ ]$ one defines *circular Hilbert transformation*, $\tilde{H}$ (see Section 2), and defines $\tilde{A}f = f + \ \tilde{H}f$ to be the associated analytic signal, that is the boundary value of an analytic function inside the unit disc. Analytic signals in the two contexts can be further written in the complex-valued amplitude–frequency modulation form $\ (\ )e^{\ (\ )}$, where $\ (\ ) \geqslant 0$ is the *instantaneous amplitude*, $\ (\ )$ is the *instantaneous phase*, and $\ '(\ )$ is the *instantaneous frequency* of the original real-valued signal f. In such a way Hilbert transformations in the two contexts play important roles in time–frequency analysis. We note that the operator $\tilde{A}$ may also be defined for complex-valued signals in the same way. Now what is interesting is the following question: For what functions $\ (\ )$ and $\ (\ )$ is the function $\ (\ )e^{\ (\ )}$ analytic? This paper gives an answer to the question for the particular case $\ \equiv 1$.

It may be easily verified, using the property $\tilde{H}^2 = -I$ (modulo constants), where I denotes the identity operator, that if f is a real-valued signal, then $\tilde{A}^2 f = 2\tilde{A}f$. The following more general result is helpful.

Theorem 1.1. *A complex-valued signal f is an analytic signal if and only if $\tilde{A}f = 2f$* (*modulo constants*).

Proof. If f is analytic and $f = g + \ \tilde{H}g$, then

$$\tilde{A}f = (g + \ \tilde{H}g) + \ \tilde{H}(g + \ \tilde{H}g) = g - \tilde{H}^2 g + 2\ \tilde{H}g = 2(g + \ \tilde{H}g) = 2f.$$

On the other hand, if $\tilde{A}f = 2f$ and $f = g + \ \ $, then

$$(g + \ \ ) + \ \tilde{H}(g + \ \ ) = 2(g + \ \ ).$$

This reduces to

$$-\tilde{H}\ \ + \ \tilde{H}g = g + \ \ .$$

By comparing the imaginary parts of the two sides of the last relation we have $\ = \tilde{H}g$, and so $f = g + \ \tilde{H}g$, being analytic. The proof is complete. □

The periodic version of Bedrosian's theorem [1,9] asserts that if $\ (\ )$ is real-valued of low frequencies and $e^{\ (\ )}$ of high frequencies, as generally expected in practice, then $\tilde{A}(\ (\ )e^{\ (\ )}) = \ (\ )\tilde{A}e^{\ (\ )}$. The question thus reduces to finding conditions that guarantee $e^{\ (\ )}$ to be analytic, or, according to Theorem 1.1, $\tilde{A}e^{\ (\ )} = 2e^{\ (\ )}$. Should this be true, we consequently have $\tilde{A}(\ (\ )e^{\ (\ )}) = 2\ (\ )e^{\ (\ )}$. Invoking Theorem 1.1 again we obtain that the complex signal $\ (\ )e^{\ (\ )}$ is analytic. This observation shows that the cases corresponding to $\ \equiv 1$ are, in fact, of particular importance.

On the unit circle (the periodic case) the relation $\tilde{A}e^{\ (\ )} = 2e^{\ (\ )}$ is equivalent to

$$\tilde{H}\cos\ (\ ) = \sin\ (\ ) \quad \text{and} \quad \tilde{H}\sin\ (\ ) = -\cos\ (\ ).$$

On the real line the counterpart relation is equivalent to

$$H\cos\ (\ ) = \sin\ (\ ) \quad \text{and} \quad H\sin\ (\ ) = -\cos\ (\ ).$$

In virtue of the Cayley transformation, theories on the upper-half complex plane and on the unit disc may be transferred to each other. Some observations and examples along this line were first made in [9]. This paper devotes to the theoretical aspect that is independent of [9]. In below we first give an answer to the question in the unit circle context. We show that a sufficient and necessary condition for functions $\ (\ )$ to have the desired property is that $d\ (\ )$ is a harmonic measure on the unit circle. With the Cayley transformation this induces the analogous result on the real line. These results lead to the theory of Hardy-space-preserving weighted trigonometric series and weighted Fourier transformations in the respective contexts.

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2. Harmonic measure on the unit circle

We will be working on the complex plane $\mathbf{C}$. Denote by $\mathbf{D}$ the unit disc centered at the origin, and by $\partial\mathbf{D}$ its boundary. The circle $\partial\mathbf{D}$ has its canonical parametrization via the mapping $\ = e\ \ , 0 \leqslant\ \ \leqslant 2\ $. For a function $f \in L^2([0, 2\ \ ])$ we have

$$f(\ ) = \sum_{=-\infty}^{\infty} c\ e\ \ ,$$

where $c\ $'s are the Fourier coefficients of f and the convergence is in both the L^2-sense and the a.e. pointwise sense (Carleson's theorem). Since the coefficients $c\ $'s are bounded, the related series $\sum_{=0}^{\infty} c\ \ \ $ and $\sum_{=-\infty}^{-1} c\ \ \ $ converge to analytic functions, denoted by f^+ and f^-, in $\mathbf{D}$ and in $\mathbf{C} \setminus \bar{\mathbf{D}}$, respectively. Detailed analysis shows that the L^2-limit or pointwise limit functions $\sum_{=0}^{\infty} c\ e\ \ $ and $\sum_{=-\infty}^{-1} c\ e\ \ $ are respectively the boundary values of f^+ and f^-. Without ambiguity those boundary values are still denoted by f^+ and f^-. The boundary functions f^+ and f^- are Hardy H^2-space components of f, inside $\mathbf{D}$ and outside $\bar{\mathbf{D}}$, respectively. Define *circular Hilbert transform* of square integrable functions on the circle by

$$\tilde{H} f(\ ) = -\ \ \sum_{=-\infty}^{\infty} \operatorname{sgn}(\ ) c\ e\ \ , \tag{1}$$

where sgn is the signum function taking values 1, -1 or 0 for

Circular Hilbert transformation has a singular integral representation:

$$\tilde{H} f(\ ) = \frac{1}{2}\text{p.v.} \int_{-} \cot\left(\frac{\ -\ }{2}\right) f(\ )\, d\ , \quad \text{a.e.} \tag{4}$$

Some basic knowledge of the Hardy spaces in the unit disc, H , $0 <\ \leqslant \infty$, and of inner functions will be used for which we refer the reader to [10] or [2]. In below the notation $|E|$ for a given measurable set E stands for the Lebesgue measure of the set E.

Theorem 2.1. *Assume that is a continuous and strictly increasing function on* $[0, 2\ ]$ *with* $|\ ([0, 2\ ])| = 2$. *Then the following two conditions are equivalent.*

(i) $d\ (\ )$ *is a harmonic measure on the unit circle.*

(ii) $$\tilde{H} \cos\ (\ ) = \sin\ (\ ) \quad \textit{and} \quad \tilde{H} \sin\ (\ ) = -\cos\ (\ ) - a \tag{5}$$
for some $a \in \mathbf{D}$.

Proof. (i) $\to$ (ii) A harmonic measure is associated with a Möbius transform $_a(\ ) = (\ - a)/(1 - \bar{a}\ )$, $a \in \mathbf{D}$, and we may write $(\ ) = \ _a(\ )$, where $_a$ is defined through $_a(e\ ) = e^{\ _a(\ )}$. It may be easily computed (or see [2]) that

$$\frac{1}{2}\frac{d\ _a(\ )}{d} = \frac{1}{2}\frac{1 - |a|^2}{1 - 2|a|\cos(\ -\ _a) + |a|^2} = \ _a(\ ) > 0, \tag{6}$$

where $a = |a|e^{\ _a}$, and $_a$ is the Poisson kernel for the point $a \in \mathbf{D}$.

Note that $_a$ is an analytic function in a neighborhood of $\bar{\mathbf{D}}$. By invoking the relation (2) for $f = f^+ = \ _a$, we have

$$\tilde{H}\ _a = \ _a - \ _a(0).$$

Due to the fact $_a(0) = -a$, the last relation reads

$$\tilde{H} e^{\ _a(\ )} = e^{\ _a(\ )} + a,$$

or

$$\tilde{H} \cos\ _a(\ ) = \sin\ _a(\ ) \quad \text{and} \quad \tilde{H} \sin\ _a(\ ) = -\cos\ _a(\ ) - a.$$

(ii) $\to$ (i) The assumptions on imply that $e^{\ (\ )} \in L^2(\partial\mathbf{D})$ and

$$\tilde{H} e^{\ (\ )} = e^{\ (\ )} + a.$$

So,

$$e^{\ (\ )} + \ \tilde{H} e^{\ (\ )} = 2e^{\ (\ )} + a. \tag{7}$$

The left-hand side of (7), due to (2), is equal to

$$2(e^{\ (\ )})^+ - c_0, \tag{8}$$

where we denote by $(e^{\ (\ )})^+$ the Hardy space projection of $e^{\ (\ )}$, and c_0 the constant term of the Fourier expansion of $e^{\ (\ )}$. On the other hand, the right-hand side of (7) is equal to

$$2(e$$

Comparing (8) and (9), we have

$$-c_0 = (2e^{\ \ (\ )})^- + a.$$

Therefore,

$$c_0 = -a \quad \text{and} \quad (2e^{\ \ (\ )})^- = 0.$$

The last two relations show that $e^{\ \ (\ )}$ itself is the boundary value of an analytic function, f, in $\mathbf{D}$, with $f(0) = -a$.

Next we show $f(\mathbf{D}) \subset \mathbf{D}$. First, $f \in H^2$ and $f|_{\partial\mathbf{D}}(\ ) = e^{\ \ (\ )} \in L^\infty$ imply $f \in H^\infty$ [2]. Since $f|_{\partial D}$ is unimodular, we obtain that f is an inner function. From the factorization theorem of inner functions we have $f = cBS$, where c is a constant with $|c| = 1$, B is a Blaschke product and S a singular function [2]. The fact that Möbius transforms map $\mathbf{D}$ into $\mathbf{D}$ implies $B(\mathbf{D}) \subset \mathbf{D}$. As for any singular function S we have

$$\log\bigl|S(\ )\bigr| = -\int P\ (\ )\,d\ \ (\ ) < 0,$$

where $d\ $ is a nonnegative Borel measure. Therefore $S(\mathbf{D}) \subset \mathbf{D}$, and thus $f(\mathbf{D}) \subset \mathbf{D}$.

Now we show that f is bijective from $\mathbf{D}$ to $\mathbf{D}$. Since f is an inner function, it is the Poisson integral of its boundary value $e^{\ \ (\ )}$ [2, Chapter 3, Corollary 3.2]. This, together with the fact that the boundary value is continuous, implies that f is continuously extended to the closure of the unit disc $\mathbf{D} \cup \partial\mathbf{D}$. From this we obtain that f only has at most finitely many zeros in $\mathbf{D}$, for if there were infinitely many zeros then there should exist cluster points which were either inside $\mathbf{D}$ or on the boundary $\partial\mathbf{D}$. Both, however, are impossible due to the unimodularity. The assumptions of the theorem then imply that the circle $\partial\mathbf{D}$ is the continuous and injective image of itself with the same orientation. The Argument Principle may be extended to conclude that the analytic mapping $f : \mathbf{D} \to \mathbf{D}$ is bijective, and so $f \in \mathrm{Aut}(\mathbf{D})$, the analytic automorphic group of $\mathbf{D}$, and thus f is a Möbius transform [3]. Since $f(0) = -a$, for some $\ _1 \in \mathbf{R}$, it is of the form

$$f(\ ) = e^{\ \ 1}\ _{ae^{-\ \ 1}}(\ ),$$

and therefore

$$e^{\ \ (\ )} = e^{\ (\ 1+\ \ _{ae^{-\ \ 1}}(\ ))}.$$

Consequently,

$$d\ \ (\ ) = d\ \ _{ae^{-\ \ 1}}(\ ),$$

being a harmonic measure. The proof is complete. □

The functions $\ _a$ may be continuously extended to the real line modulo the condition $\ _a(\ + 2\ \ ) = \ _a(\ ) + 2\ $. The extended functions have continuous $2\ $-periodic derivatives $\tilde{\ }_a(\ )$. Corresponding to $\ _a$, the period functions $e^{\ \ \ _a}$, $\ > 0$, except for the trivial case $a = 0$ corresponding to $e^{\ \ }$, are not included in the general form of Picinbono [7], of which the derivatives of the phase functions are not periodic. The single component case of Picinbono's phase functions coincides with what is studied in Section 4, of which the derivatives of the phases are the Poisson kernels of the real line.

Theorem 2.2. *With the periodic extensions of* $\cos\ _a(\ )$ *and* $\sin\ _a(\ )$ *to the real line, we have*

$$H\cos\ _a(\ ) = \sin\ _a(\ ) \quad \textit{and} \quad H\sin\ _a(\ ) = -\cos\ _a(\ ). \tag{10}$$

Proof. First we note that in the principal value integral sense the Hilbert transforms are well defined for the oscillatory functions $\cos\ _a(\ )$ and $\sin\ _a(\ )$. Using the identity (see [8])

$$\lim_{N\to\infty} \sum_{=-N}^{N} \frac{1}{-\ \ +2\ } = \frac{1}{2}\cot\left(\frac{-}{2}\right),$$

we have

$$\begin{aligned} \text{p.v.}\frac{1}{\ }\int_{-\infty}^{\infty} \frac{1}{-}\cos\ _a(\ )\,d\ &= \text{p.v.}\frac{1}{\ }\int_0^{2} \sum_{=-\infty}^{\infty} \frac{1}{-\ \ +2\ }\cos\ _a(\ )\,d \\ &= \text{p.v.}\frac{1}{2}\int_0^{2} \cot\left(\frac{-}{2}\right)\cos\ _a(\ )\,d\ = \tilde{H}\cos\ _a(\ ). \end{aligned}$$

The desired relation for $\cos\ _a(\ )$ then follows from Theorem 2.1. The assertion for $\sin\ _a(\ )$ may be proved similarly. □

3. Analytic weighted trigonometric systems

As shown in Section 2, every complex number $a \in \mathbf{D}$ is associated with a Möbius transform $\ _a$, and correspondingly a Poisson kernel $\ _a(\ )$. The function $\ _a$ defined from the relation $e^{\ _a(\ )} = \ _a(e\ )$ satisfies $\ '_a(\ ) = \ _a(\ )$. Writing $a = |a|e^{\ _a}$, it is easy

We have the following

Theorem 3.1. *Let $a \in \mathbf{D}$ and $\mathcal{F}_a = \{\frac{1}{\sqrt{2}} e \quad {}_a(\ )\}^{\infty}_{=-\infty}$, the corresponding weighted trigonometric system. Then*

(i) *$\mathcal{F}_a$ is an orthonormal system in $L^2_a(\partial\mathbf{D})$.*
(ii) *The Plancherel theorem holds for the system. In particular, the system is complete in $L^2_a(\partial\mathbf{D})$.*
(iii) *Carleson's theorem holds with respect to the system $\mathcal{F}_a$.*
(iv) *The mapping ${}_a(\ )$ preserves the Hardy spaces inside and outside the unit circle.*

Proof. The assertion (iv) follows from the conformal mapping property of Möbius transform. The assertions (i) to (iii) are proved via change of variable $\ = {}_a(\ )$, as shown in the following.

(i) Setting $e^a(\ ) = \frac{1}{\sqrt{2}} e \quad {}_a(\ )$, we have

$$\left\langle e^a, e^a \right\rangle_a = \frac{1}{2} \int_0^{2} e^{\ (\ -\ )\ {}_a(\ )} \quad {}_a(\ )\, d \ = \frac{1}{2} \int_0^{2} e^{\ (\ -\ )} \ d \ = \qquad ,$$

the Kronecker delta function.

(ii) For any function $f \in L^2_a(\partial\mathbf{D})$, denote by $c^a(f)$ the th Fourier coefficient of f with respect to the weighted trigonometric system:

$$c^a(f) = \left\langle f, e^a \right\rangle_a.$$

Through the change of variable it is easy to verify

$$c^a(f) = c^0(F),$$

where $c^0(F)$'s are the standard Fourier coefficients of the function $F(\ ) = f(\ {}_a^{-1}(\ )) \in L^2(\partial\mathbf{D})$. Since $\|F\|_0 = \|f\|_a$, the classical Plancherel theorem for F implies

$$\|f\|_a^2 = \sum \left| c^a(f) \right|^2.$$

(iii) Carleson's theorem asserts that

$$\lim_{N\to\infty} \sum_{=-N}^{N} c^0(F) e \qquad = F(\ ), \quad \text{a.e.}$$

Since $c^0(F) = c^a(f)$, $\ = {}_a(\ )$ and $F(\ ) = f(\ )$, we obtain

$$\lim_{N\to\infty} \sum_{=-N}^{N} c^a(f) e \quad {}^{a(\ )} = f(\ ), \quad \text{a.e.}$$

The proof is complete. □

For different a the shapes of $\cos\ {}_a(\ )$ (also those of $\sin\ {}_a(\ )$) are different (see examples in [9]). It is observed that the closer $|a|$ gets to 1, the sharper the graph of $\cos\ {}_a(\ )$

is. The weighted trigonometric systems are expected to be better suitable and adaptable, along with different choices of a, to nonlinear and nonstationary time–frequency analysis.

4. Counterpart results on the real line

Hilbert transformation on the real line will be taken to be of the distributional sense [4,5]: If $F(\ ) = \ (\ ,\ ) + \ (\ ,\ )$ is an analytic function in the upper-half complex plane, and and are respectively the *harmonic representations* of distributions S and T on the real line, then we say that T is a Hilbert transform of S, denoted by $HS = T$. This definition, in particular, implies that if $HS = T$, then $HS = T + c$ for any constant c. There will be no ambiguity arising out of this: When we have $HS = T$, it means that T is a representative among all the Hilbert transforms of S. Based on this definition, it can be proved that any distribution has a Hilbert transform, and, in particular, any bounded measurable function has a Hilbert transform. Note that the above definition coincides with the standard definition of Hilbert transformation for functions in good function classes, such as in $L\ (\mathbf{R})$.

The Cayley transformation

$$(\ ) = \frac{-}{+}$$

conformally maps the upper-half complex plane to the disc $\mathbf{D}$. It maps the real line to the unit circle through

$$(\ ) = \frac{1 - \ ^2}{1 + \ ^2} + \ \frac{2}{1 + \ ^2}.$$

Setting $\ = 2\tan^{-1}\ $, the above reads $(\ ) = \cos\ + \ \sin\ $, where $\in (-\ ,\ )$, $\in (-\infty, \infty)$. Now, if $f(\ ) = \cos\ (\ ) + \ \sin\ (\ )$ is the boundary value of an analytic function inside $\mathbf{D}$, then

$$F(\ ) = \cos\ (2\tan^{-1}\ ) + \ \sin\ (2\tan^{-1}\ )$$

is the boundary value of the image analytic function in the upper-half plane under the

Moreover, if (i) *and* (ii) *hold, then* $(\ ) = A + {}_a(2\tan^{-1}\ )$ *for some* $A \in \mathbf{R}$ *and* $a \in \mathbf{D}$.

Proof. It may be shown that with $(\ ) = A + {}_a(2\tan^{-1}\ )$, ${}'_a(\ )$ is a Poisson kernel on the circle if and only if $'(\ )$ is a Poisson kernel on the line. It is, in fact, a bijective mapping between all Poisson kernels on the circle and all those on the line. The precise correspondence reads

$$\frac{1}{2}\frac{d}{d}\ {}_a(2\tan^{-1}\ ) = \frac{1}{}\frac{a}{(\ -\ a)^2 + \ a^2} = P_{\ a}(\ -\ a),$$

where $_a = \frac{1-|a|^2}{1+2|a|\cos\ _a+|a|^2}$, $_a = \frac{2|a|\sin\ _a}{1+2|a|\cos\ _a+|a|^2}$ and $a = |a|e^{\ a}$.

Next we point out that, from the distributional definition of the Hilbert transform and the properties of the Cayley transformation, the assertion (ii) of the theorem is equivalent to the assertion (ii) of Theorem 2.1. The proof is thus complete when invoking Theorem 2.1. □

It is a property of the harmonic measures that the mappings $(\ )$ in Theorem 4.1 map the Hardy H -spaces on the unit circle to the weight Hardy H -spaces on the real line with the weights $'(\ )^{1/}$, $0 < \ \leqslant \infty$. We refer the reader to [2].

5. Weighted Fourier transformation on the line

Parallel to Section 3 we can formulate a weighted Fourier transformation theory.

For $a \in \mathbf{D}$, define

$$L^2_a(\mathbf{R}) = \left\{ f : \mathbf{R} \to \mathbf{C} \;\middle|\; \left(\int_{-\infty}^{\infty} \left|f(\ )\right|^2 \tilde{\ }_a(\ )\,d\ \right)^{1/2} < \infty \right\}, \tag{14}$$

where $\tilde{\ }_a$ is the 2 -periodization of the Poisson kernel $_a$ on the circle.

Denote by

$$\|f\|_a = \left(\int_{-\infty}^{\infty} \left|f(\ )\right|^2 \tilde{\ }_a(\ )\,d\ \right)^{1/2}$$

the norm of $f \in L^2_a(\mathbf{R})$.

The space $L^2_a(\mathbf{R})$ forms a Hilbert space under the inner product

$$\langle f, g\rangle_a = \int_{-\infty}^{\infty} f(\ )\overline{g(\ )}\,\tilde{\ }_a(\ )\,d\ .$$

Note that if $a = 0$, then all just defined reduce to the standard case on $\mathbf{R}$.

Define the associated weighted Fourier transformation by

$$F_a(f)(\ ) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\ \ _a(\ )} f(\ )\,\tilde{\ }_a(\ )\,d\ .$$

Similarly to the series case studied in Section 3, we have the Plancherel theorem and the corresponding Fourier inversion theorem that all reduce to the standard case through the change of variable. We omit the details. We shall, however, cite below the corresponding Poisson summation formula.

For $f \in L_a^2(\mathbf{R})$ set

$$\tilde{f}(\ ) = \sum_{=-\infty}^{\infty} f(\ + 2\ \ ).$$

Then $\tilde{f} \in L_a^2(\partial\mathbf{D})$. In both the L^2-convergence and the pointwise convergence sense,

$$\tilde{f}(\ ) = \sum c^a e^{\ \ a(\ )}.$$

We shall show that

$$F_a(f)(\ ) = c^a, \qquad \text{ is any integer.} \tag{15}$$

Taking the relation (15) for granted for the moment, we have

$$\tilde{f}(\ ) = \sum F_a(f)(\ )e^{\ \ a(\ )}.$$

If, in particular, taking $\ = \ _0$ such that $e^{\ \ 0} =)$

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