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THE DIFFERENTIAL INTEGRAL EQUATIONS ON SMOTH CLOSED ORIENTABLE MANIFOLDS

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Abstract Using integration by parts and Stokes' formula the authors give a new definition of Hadamard principal value of higher order singular integrals with Bochner-Martinelli kernel on smooth closed orientable manifolds in $\mathbf{C}^n$. The Plemelj formula and composite formula of higher order singular integral are obtained. Differential integral equations on smooth closed orientable manifolds are treated by using the composite formula.

Key words Bochner-Martinelli kernel, Plemelj formula, Composite formula, Higher order singular integral, Differential integral Equation

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1 Introduction

nce e o z e e o o C c ype n e
Boc ne M ne e ne p o ed n [1] een cce y ed o dy
o n n e e on o n $\overline{\partial}_b$

ed ce o de o n y o o de n n e nd e n y e C c y n n e o e p e o de n n e d d p nc p e e o e e e p e ed y C c y p nc p e

o p c y e con de o de n n e on o nd y o o nded do n D oo o nd y n co p e p ne

$$\overline{2\pi i}\int_{\partial D}\frac{f\ \zeta}{\zeta - z\ ^{2}}d\zeta, \quad z\in D,$$

e $f\ \zeta\ \in H_1\ \alpha$

n o de o dy o de n n e Boc ne M ne e ne y e dy o o n o de Boc ne M ne ype n e

$$\int_{\partial D} f\ \zeta\ \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^2} K\ \zeta, z\ , \qquad z \in \mathbf{C}^n \setminus \partial D,$$

e f d e en e nd e on o H_1 α on ∂D. e e

Lemma 1 Le f e d e en e nd e on o H_1 α on $\overline{D}$ n o ny $w \in \partial D$ nd ny $B\ w, \epsilon$ o d ϵ ϵ c en po e n e cen ed $w \in \partial D$ e e

$$\int_{\partial D \setminus B(\omega,\epsilon)} f\ \zeta\ \frac{\overline{\xi}_k - \overline{\omega}_k}{|\zeta - \omega|^2} K\ \zeta, \omega$$

$$- \frac{}{n} C_n \int_{\partial(\partial D \setminus B(\omega,\epsilon))} f\ \xi$$

$$\cdot \frac{\sum_{j=1}^{n} -\ {}^{j-1}\ \overline{\xi}_j - \overline{\omega}_j\ -\ {}^{k-1} d\zeta_1 \wedge \cdots \wedge\ d\zeta_k\ \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge\ d\overline{\zeta}_j\ \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - \omega|^{2n}}$$

$$\frac{}{n} C_n \int_{\partial D \setminus B(\omega,\epsilon)} df\ \zeta$$

$$\cdot \frac{\sum_{j=1}^{n} -\ {}^{j-1}\ \overline{\xi}_j - \overline{\omega}_j\ -\ {}^{k-1} d\zeta_1 \wedge \cdots \wedge\ d\zeta_k\ \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge\ d\overline{\zeta}_j\ \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - \omega|^{2n}},$$

e C_n $\frac{(n-1)!}{(2\pi i)^n}$

Proof By n e on y p e e

$$\int_{\partial D \setminus B(\omega,\epsilon)} f(\zeta) \frac{\overline{\zeta}_k - \overline{\omega}_k}{|\zeta - \omega - -|\cdot|-|------|-\cdot\wedge\wedge\wedge-\cdot\wedge-\wedge\wedge--|---||--------\cdot\wedge\wedge\cdot\wedge-\cdot\wedge-\wedge\wedge--|--}$$

f(f))

∂D B(ω,)

iζ— 51288]TJ /R37(

$$PV \frac{}{n} C_n \int_{\partial D} df \; \zeta$$
$$\cdot \frac{\sum_{j=1}^{n} - \;\; {}^{j-1} \; \overline{\zeta}_j - \overline{\omega}_j \;\; - \;\; {}^{k-1} d\zeta_1 \wedge \cdots \wedge \; d\zeta_k \;\; \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge \; d\overline{\zeta}_j \;\; \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - \omega|^{2n}}.$$

o y o e de n on o d d p nc p e p e nd c e e n n de n on o e d n e d c ded d e en p o o de n n e d ec y nd on y ep n e p n y e c n o d co p c ed c c on n pp c on 2 de ned ene po n $\omega \in \partial D$ nd d d p nc p e n e o C c y p nc p e o e c n ze e o C c y p nc p e d ec y

A e p e y e e o o C c y n n e on ∂D *see* – e e

Theorem 1 e e o o o de n n e nde p on n e n on z pp o c $\omega \in \partial D$ o nne p nd o e p o D n o o de Boc ne M ne ype n e

$$F \; z \qquad \int_{\partial D} f \; \zeta \; \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^2} K \; \zeta, z \qquad z \in \mathbf{C}^n \setminus \partial D$$

o o e e o

$$F_i \; \omega \qquad FP \int_{\partial D} f \; \zeta \; \frac{\overline{\zeta}_k - \overline{\omega}_k}{|\zeta - \omega|^2} K \; \zeta, z \qquad \frac{}{2n} \left[\frac{\partial f}{\partial \omega_k} \; \omega \qquad - \;\; {}^n \frac{\partial f}{\partial \omega_1} \; \omega \; \right],$$
$$F_e \; \omega \qquad FP \int_{\partial D} f \; \zeta \; \frac{\overline{\zeta}_k - \overline{\omega}_k}{|\zeta - \omega|^2} K \; \zeta, z \; - \frac{}{2n} \left[\frac{\partial f}{\partial \omega_k} \; \omega \qquad - \;\; {}^n \frac{\partial f}{\partial \omega_1} \; \omega \; \right].$$

Proof By n e on y p nd o e o e e

$$F(z) = \int_{\partial D} f(\zeta) \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^2} K(\zeta, z)$$
$$= -\frac{1}{n} C_n \int_{\partial D} f(\zeta)$$
$$\cdot d \left[\frac{\sum_{j=1}^{n} (-1)^{j-1} (\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}} \right]$$
$$= -\frac{1}{n} C_n \int_{\partial D}$$
$$\cdot d \left[f(\zeta) \frac{\sum_{j=1}^{n} (-1)^{j-1} (\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}} \right]$$
$$+ \frac{1}{n} C_n \int_{\partial D} df(\zeta)$$
$$\cdot \frac{\sum_{j=1}^{n} (-1)^{j-1} (\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}}$$
$$= \frac{1}{n} C_n \int_{\partial D} df(\zeta)$$
$$\cdot \frac{\sum_{j=1}^{n} (-1)^{j-1} (\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}}.$$

e o e

$$F_i(\omega) = \lim_{z\to\omega^+} \frac{1}{n} C_n \int_{\partial D} df(\zeta)$$

$$\cdot \frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}}$$

$$= \lim_{z\to\omega^+} \frac{1}{n} \int_{\partial D} \frac{\partial f}{\partial \zeta_k}(\zeta) K(\zeta, z) + (-1)^{n-1} \frac{1}{n} C_n \lim_{z\to\omega^+} \int_{\partial D} \frac{1}{|\zeta - z|^{2n}}$$

$$\cdot \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} (\overline{\zeta}_j - \overline{z}_j) d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_n.$$

n o

$$_{z\to\omega^+} - \quad ^{n-1} C_n \int_{\partial D} \overline{|\zeta - z|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{\zeta}_j} \; \overline{\zeta}_j - \overline{\zeta}_j \quad - \quad ^{k-1} d\zeta_1$$

$$\wedge \cdots \wedge \; d\zeta_k \; \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_n$$

$$- \quad ^{n-1} C_n PV \int_{\partial D} \overline{|\zeta - \omega|^{2n}} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{\zeta}_j} \; \overline{\zeta}_j - \overline{\omega}_j \quad - \quad ^{k-1} d\zeta_1$$

$$\wedge \cdots \wedge \; d\zeta_k \; \wedge \cdots \wedge d\zeta_n$$

Theorem 2 Co po e o o o de n n e ppo e $\phi \xi$ o o p c n ne o od o ∂D $\dfrac{\partial \phi \xi}{\partial \overline{\xi}}$ n $U \partial D$ n co po e o

$$\int_{\partial D_\eta} K \ \zeta,\eta \int_{\partial D_\xi} \phi \ \xi \ \frac{\overline{\eta}_k - \overline{\zeta}_k}{|\eta-\xi|^2} K \ \eta,\xi \qquad - \frac{\partial \phi \ \zeta}{\partial \zeta_k}$$

d n no on

$$S\phi \quad 2\int_{\partial D_\xi} \phi \ \xi \ K \ \eta,\xi$$

$$S_1\phi \quad 2\int_{\partial D_{xi}} \phi \ \xi \ K_1 \ \eta,\xi \ , \qquad K_1 \ \eta,\xi \quad \frac{\overline{\eta}_k - \overline{\xi}_k}{|\eta-\xi|^2} K \ \eta,\xi \ ,$$

co po e o c n e en

$$SS_1\phi \qquad \int_{\partial D_\eta} K \ \zeta,\eta \int_{\partial D_\xi} \phi \ \xi \ K_1 \ \eta,\xi \qquad \frac{\ }{n} \frac{\partial \phi \ \zeta}{\partial \zeta_k}.$$

Proof p on $\dfrac{\partial \phi \ \zeta}{\partial \overline{\zeta}}$ n $U \partial D$ p e $\phi \ \zeta$ c n e o o p c y e ended n o D ence co po e o 2 o $\dfrac{\partial \phi \ \zeta}{\partial \zeta_k}$ d n o e n on e e

$$\begin{aligned}
&\int_{\partial D_\eta} K(\zeta,\eta) \int_{\partial D_\xi} \phi(\xi) \frac{\overline{\eta}_k - \overline{\xi}_k}{|\eta-\xi|^2} K(\eta,\xi) \\
&= \int_{\partial D_\eta} K(\zeta,\eta) \int_{\partial D_\xi} \frac{1}{n} C_n d\phi(\xi) \cdot \\
&\cdot \frac{\sum_{j=1}^n (-1)^{j-1}(-1)^{k-1}(\overline{\xi}_j - \overline{\eta}_j) d\xi_1 \wedge \cdots \wedge [d\xi_k] \wedge \cdots \wedge d\xi_n d\overline{\xi}_1 \wedge \cdots \wedge [d\overline{\xi}_k] \wedge \cdots \wedge d\overline{\xi}_n}{|\eta-\xi|^{2n}} \\
&= \frac{1}{n} \int_{\partial D_\eta} K(\zeta,\eta) \int_{\partial D_\xi} C_n \frac{\partial \phi(\xi)}{\partial \xi_k} \frac{\sum_{j=1}^n (-1)^{j-1}(\overline{\xi}_j - \overline{\eta}_j) d_{\xi_1} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge [d\overline{\xi}_k] \wedge \cdots \wedge d\overline{\xi}_n}{|\eta-\xi|^{2n}} \\
&= \frac{1}{n} \int_{\partial D_\eta} K(\zeta,\eta) \int_{\partial D_\xi} \frac{\partial \phi(\xi)}{\partial \xi_k} K(\eta,\xi) = \frac{1}{4n} \frac{\partial \phi(\zeta)}{\partial \zeta_k}.
\end{aligned}$$

p oo co p e e

Remark A e D co p e nd ϕ $P_{s,t}$ o eneo on c po yno o de ee s e pec o z nd de ee t e pec o $\overline{z}$ n o $n >$ co po e o 2 n c e o C c y n n e c nno d n e n $P_{s,t}$ o o p c po yno e n t o n Le 2 e e $\phi \ \zeta \in C^{(1)} \ \partial D$ nd c n e o o p c y e ended n o D L e e n o e 2 e e $\phi \ \xi$ o o p c n ne o od o ∂D c n e cenn n e co po 2

Z

$$_1\phi \quad 2 \quad _{\partial D_\xi} \phi \ \xi \ K_1 \ \eta \ \xi$$

App y n ope o

$$S\phi \quad 2\int_{\partial D_\xi} \phi \ \xi \ K \ \eta,\xi$$

o o de o e e

$$SS_1\phi \quad \frac{}{n}\frac{\partial\phi \ \zeta}{\partial\zeta_k} \quad S\psi \quad \int_{\partial D_\eta} \psi \ \eta \ K \ \zeta,\eta \ ,$$

nde e o nd y cond on e c n n e y o e o e p d e en e on o ϕ.

4 Higher Order Singular Integral Equations and Partial Differential Integral Equations

n ec on e d c ne p ce L c con o co p e e d e en e nc on e p de e y o de cond on on ∂D. n c o de Boc ne M ne n n e nd ce ne ope o on L n o o n e o e o de n n e e on Boc ne M ne e ne

e con de o de n n e e on

$$aS\phi \quad bS_1\phi \quad T\phi \quad \psi,$$

e a, b e co p e con n $\psi \in L$ en nc on $T\phi$ $\int_{\partial D_\xi} \phi \ \xi \ L \ \eta,\xi$ e e ne $L \ \eta,\xi$ co p e e e o d e en o o de ee $2n-$ nd $L \ \eta,\xi \ \in H_1$ e o de p de e o $L \ \eta,\xi$ e pec o η nd ξ y o de cond on on ∂D

y e con de c c e c e on o

$$aS\phi \quad bS_1\phi \quad \psi. \qquad 2$$

ndeed y e n on d e en nd n n e e on

App y n ope o

$$M \quad aI - bS, \qquad 2$$

e

$$I\phi \quad \phi \qquad \phi \in L \ , \qquad 22$$

o o de o 2 nd z n co po e o e e

$$a\phi \quad \frac{b}{n}\frac{\partial\phi \ \zeta}{\partial\zeta_k} \quad S\psi. \qquad 2$$

o n p d e en e on nde e o nd y e cond on e c n o n n no n nc on ϕ n e y

ene e on c n e ed ced o e en ed e on y n ope o M. n c pp y n ope o M o e on nd n co po e o e e

$$a\phi \quad \frac{b}{n}\frac{\partial\phi \ \zeta}{\partial\zeta_k} \quad ST\phi \quad S\psi. \qquad 2$$

App y n oe o e c n n o de o n e on o n n e nd o d n y n e 2 e e

$$ST\phi \quad \int_{\partial D_\eta} K \ \zeta,\eta \ \int_{\partial D_\xi} \phi \ \xi \ L \ \eta,\xi$$

$$\int_{\partial D_\xi} \phi \ \xi \ \int_{\partial D_\eta} L \ \eta,\xi \ K \ \zeta,\eta \ .$$

e c n n p o e $\int_{\partial D_\eta} L \ \eta,\xi \ K \ \zeta,\eta$ e o de cond on e pec o ξ nd ζ 2 p 2 2 e o e 2 ed e on

on 2 nd e e en o e need on y p o e o on o 2 e e on n c pp y n ope o

$$M^* \quad aI \quad bS \qquad 2$$

o o de o 2 nd e co po e o e o n ed e y o e e e

o e ppo e n e on a,b e co p e con n $\psi \in L$ nd e ne $L \ \eta,\xi$ o ope o T e on o H_1, n

o on o c ce ce on 2 o c n e o ned n e y n L y o n p d e en e on 2 nde e o nd y e cond on n L o de n n e e on e en o ed e on 2

Acknowledgment econd o e p e nce e de o n e y o Ne n nd nd o e o o n o p y nd ood o n cond on p o ded d n y n A d e n A

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