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Properties of Poisson Kernel for a Degenerate Elliptic Equation

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The solution of the Dirichlet problem inside the unit sphere for equation (2) is given by the Poisson formula (see [3])

$$U(x) = \frac{1}{\omega_n} \int_{S^{n-1}} P(x, v) f(v) \, \mathrm{d}v, \tag{4}$$

where $\omega_n = |S^{n-1}|$ is the area of the unit sphere in $\mathbb{R}^n$ and $\mathrm{d}v$ the area element on the unit sphere. Equation (2) can be rewritten as

$$(1 - |x|^2) \left[(1 - |x|^2) \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} + 2(n-2) \sum_{i=1}^{n} x_i \frac{\partial U}{\partial x_i} \right] = 0.$$

Replacing the constant $2(n-2)$ by an arbitrary real number τ in the last equation, we obtain (1). Thus equation (1) is a generalization of equation (2).

We provide a brief introduction to the geometric background of equation (2). It is known [2, 3] that equation (2) is in the real form of the simplest classical domain, viz. the n-dimensional unit ball, admitting a transitive group generated by rotations and non-Euclidean translations given by the real form n-dimensional Möbius transformations (5). Equation (2) is invariant not only under transformations of the transitive group, but also under reflections about the unit sphere.

The real form n-dimensional Möbius transformations are (see [3])

$$y = \frac{x - a - xx'a + x(2a'a - aa'I)}{1 - 2ax' + aa'xx'}, \tag{5}$$

where $y = (y_1, \ldots, y_n)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x' = \mathrm{col}(x_1, \ldots, x_n)$ is the transpose of x, I is the unit matrix and $a = (a_1, \ldots, a_n)$.

It can be verified that transformation (5) transforms the open unit ball $\{|x| < 1\}$ onto the open unit ball $\{|y| < 1\}$, the unit sphere $\{|x| = 1\}$ onto the unit sphere $\{|y| = 1\}$, and the point $x = a \in \{|x| < 1\}$ to the zero point $y = 0$. For $n = 2$, we may write $w = y_1 + iy_2$, $z = x_1 + \mathrm{i}x_2$ and $b = a_1 + \mathrm{i}a_2$ instead of $y = (y_1, y_2)$, $x = (x_1, x_2)$ and $a = (a_1, a_2)$. Transformation (5) then reduces to

$$w = \frac{z - b - bz\bar{z} + b^2\bar{z}}{1 - \bar{b}z - b\bar{z} + b\bar{b}z\bar{z}} = \frac{z - b}{1 - \bar{b}z}$$

which is just the usual Möbius transformation in the setting of one complex variable.

Owing to the reflection invariant property the solutions of (2) enjoy the symmetric principle and so we can extend solutions inside the unit ball to solutions outside the unit ball with the same boundary values (see [4]).

In the case $n = 2$ Equation (2) reduces to the two-dimensional Laplace equation

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = 0$$

which is invariant under the transformations of the transitive group, as well as reflections about the unit circle. On the other hand, we note that the usual

n-dimensional Laplace equation

$$\Delta U = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = 0$$

in the case of $n > 2$ does not possess the transformation invariant properties as the two-dimensional Laplacian does. For instance, the form of the n-dimensional Laplacian changes under the reflections. In this sense equation (2) is a more natural generalization of the two-dimensional Laplacian.

It is well known that in the unit ball different Poisson kernels give rise to solutions of Dirichlet problems associated with different Laplace and Laplace–Boltrami equations.

In the standard cases such as the kernel (3) and that of the usual Laplacian (see [3, 9]) one can easily show that the Poisson kernels $P(x, v) = P(\rho u, v)$ satisfy the δ-function properties, i.e. the following three properties: For $|u| = |v| = 1$, $0 \leqslant \rho < 1$,

(i) $P(\rho u, v) > 0$;
(ii) $\int_{S^{n-1}} P(\rho u, v)\, \mathrm{d}v = 1$; and
(iii) For any $\delta \in (0, 1)$, $\lim_{\rho \to 1-0} \int_{<u,v> < 1-\delta} P(\rho u, v)\, \mathrm{d}v = 0$, uniformly for $u \in S^{n-1}$.

The standard approach (e.g. see [1, 9]) to solve a Dirichlet problem is thus of the pattern: first we prove properties (i)–(iii) using the explicit expression of the Poisson kernel and then obtain solutions.

The Poisson kernel that we used to obtain the solution to the Dirichlet problem associated with the degenerate elliptic equation (1) is of the form: $\mathscr{P}(x, v) = \mathscr{P}(\rho u, v)$, $|u| = |v| = 1$, $0 \leqslant \rho < 1$, and

$$\mathscr{P}(\rho u, v) = \sum_{k=0}^{\infty} \frac{2k + n - 2}{n - 2} \rho^k \tau_k(\rho) P_k^{((n/2)-1)}(u \cdot v), \tag{6}$$

where

$$\tau_k(\rho) = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(1+\tau/2)\Gamma(k+n/2)} F\left(k - 1 + \alpha + \frac{n}{2}, k - 1 + \beta + \frac{n}{2}, k + \frac{n}{2}; \rho^2\right) \tag{7}$$

with $\alpha + \beta = 1 - k - (n + \tau)/2$ and $\alpha\beta = (\tau/4)(k + n - 2)$ (so α, $\beta \in \mathbb{R}$, $\alpha \neq \beta$), $F(a, b, c; z)$ is the hypergeometric function and

$$P_k^{(\mu)}(\xi) = \sum_{m=0}^{[k/2]} (-1)^m \frac{\Gamma(k - m + \mu)}{\Gamma(\mu) m! (k - 2m)!} (2\xi)^{k-2m} \tag{8}$$

is the Gegenbauer polynomial of degree k associated with μ (where k is an integer, $[k/2]$ is the greatest integer less than or equal to $k/2$). Our kernel function $\mathscr{P}(\rho u, v)$ does not seem to have an explicit expression in simple functions. What we want to show in this note is: in spite of the infinite series form of the kernel, we can still manage to prove properties (i)–(iii), and, as applications, obtain solutions for continuous boundary value functions which is not included in the results of [4–6]. The pattern of our approach is in the opposite order: first we obtain solutions for very smooth boundary value functions, then, by virtue of this but not of the kernel expression, we prove properties (i)–(iii), and then further obtain solutions for continuous boundary value functions.

The writing plan is as follows. In section 2 we prove Theorem 1 giving solutions to equation (1) for $C^{4n}(S^{n-1})$-boundary value functions. The theorem is a particular case of a more general result proved in [5]. The proof given in this paper does not rely on the Sobolev spaces theory and the self-adjoint operator theory in Hilbert spaces. Section 3 is devoted to proving properties (i)–(iii) of the kernel (6). As applications of properties (i)–(iii), we give solutions to equation (1) for continuous boundary value functions.

2. Solutions for $C^{4n}(S^{n-1})$-boundary value functions

Theorem 1. *For $f \in C^{4n}(S^{n-1})$ the potential formula*

$$U(x) = \frac{1}{\omega_n} \int_{S^{n-1}} \mathscr{P}(x, v) f(v) \, \mathrm{d}v$$

with the kernel function (6) *gives the solution to the equation* (1) *with the boundary value f, in the sense*

$$\lim_{\rho \to 1-0} U(\rho u) = f(u).$$

Moreover, the convergence is uniform in $|u| = 1$.

Proof. The function f is in $L^2(S^{n-1})$ and so it has the spherical Laplace eigenspace expansion:

$$f \sim \sum_{k=0}^{\infty} f_k, \tag{9}$$

where f_k is the projection of f onto the k-spherical harmonics. The Plancherel relation holds:

$$\| f \|_{L^2(S^{n-1})} = \sum_{k=0}^{\infty} \| f_k \|_{L^2(S^{n-1})}$$

(see, e.g. [7] or [8]). We now show that the right-hand side of (9) absolutely converges to f uniformly in $\{|u| = 1\}$.

Denote by $\mathscr{H}_k$ the projection operator: $f_k = \mathscr{H}_k f$, with the expression

$$f_k(u) = \mathscr{H}_k f(u) = \int_{S^{n-1}} H_k(u \cdot v) f(v) \, \mathrm{d}v,$$

where

$$H_k(\xi) = \frac{1}{\omega_n} \frac{2k + n - 2}{n - 2} P_k^{((n/2)-1)}(\xi) \quad \text{for} \quad -1 \leqslant \xi \leqslant 1.$$

For a fixed $k \in \mathbb{Z}$ and $u \in S^{n-1}$, $H_k(u \cdot v)$, as a function of $v \in S^{n-1}$, belongs to the space of the spherical harmonics of degree k, which is also the eigenspace of the operator ∂_u^2 (see below) with the eigenvalue $\lambda_{n,k} = -k(k+n-2)$.

Let ∂_u^2 be the spherical Laplacian in the polar-co-ordinate form of the usual Laplacian (see, e.g. [1])

$$\Delta = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \partial_u^2. \tag{10}$$

The operator ∂_u^2 also appears in the polar-co-ordinate form of the operator $\mathscr{L}$ (see (1)):

$$\mathscr{L} = \rho^2 \frac{\partial^2}{\partial \rho^2} + \frac{\rho}{1-\rho^2} [(n-1) + (\tau - n + 1)\rho^2] \frac{\partial}{\partial \rho} + \partial_u^2.$$

It has the form

$$\partial_u^2 = \sum_{i=1}^{n-1} \frac{1}{\sin^2 \theta_1 \cdots \sin^2 \theta_{i-1}} \frac{\partial^2}{\partial \theta_i^2} + \sum_{i=1}^{n-2} (n-1-i) \frac{\cot \theta_i}{\sin^2 \theta_1 \cdots \sin^2 \theta_{i-1}} \frac{\partial}{\partial \theta_i}$$

in the spherical system

$$u = (\cos \theta_1, \sin \theta_1 \cos \theta_2, \ldots, \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1},$$
$$\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}),$$

where

$$0 \leqslant \theta_i \leqslant \pi, \quad 1 \leqslant i \leqslant n-2, \quad 0 \leqslant \theta_{n-1} \leqslant 2\pi.$$

The operator ∂_u^2 is self-adjoint, and, for any function h in the space of k-spherical harmonics, it satisfies the characteristic equation

$$\partial_u^2 h = \lambda_{n,k} h$$

(see [3]).

Now expand $\partial_u^{2n} f$:

$$\partial_u^{2n} f \sim \sum_{k=0}^{\infty} g_k,$$

where

$$\begin{aligned} g_k(u) &= \int_{S^{n-1}} H_k(u \cdot v) [\partial_v^{2n} f(v)] \, \mathrm{d}v = \int_{S^{n-1}} [\partial_v^{2n} H_k(u \cdot v)] f(v) \, \mathrm{d}v \\ &= \lambda_{n,k}^n \int_{S^{n-1}} H_k(u \cdot v) f(v) \, \mathrm{d}v \\ &= \lambda_{n,k}^n f_k(u) \\ &= \partial_u^{2n} f_k(u). \end{aligned}$$

Then Plancherel relation for $\partial_u^{2n} f$ then concludes

$$\| \partial_u^{2n} f_k \|_{L^2(S^{n-1})} \leqslant \| \partial_u^{2n} f \|_{L^2(S^{n-1})}.$$

Since,

$$\partial_u^{2n} H_k(u\cdot v) = \lambda_{n,k}^n H_k(u\cdot v),$$

we have

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(u)| &= \sum_{k=1}^{\infty} \left| \int_{S^{n-1}} H_k(u\cdot v) f(v)\,\mathrm{d}v \right| \\ &= \sum_{k=1}^{\infty} \left| \int_{S^{n-1}} H_k(u\cdot v) f_k(v)\,\mathrm{d}v \right| \\ &= \sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^n|} \left| \int_{S^{n-1}} [\partial_v^{2n} H_k(u\cdot v)] f_k(v)\,\mathrm{d}v \right| \\ &= \sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^n|} \left| \int_{S^{n-1}} H_k(u\cdot v) [\partial_v^{2n} f_k(v)]\,\mathrm{d}v \right| \\ &\leqslant \sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^n|} \| H_k(u\cdot) \|_{L^2(S^{n-1})} \| \partial_v^{2n} f_k \|_{L^2(S^{n-1})}.\end{aligned}$$

Owing to the relation (see Chapter 2 of [2])

$$\int_{S^{n-1}} H_k(u\cdot w) H_k(v\cdot w)\,\mathrm{d}w = H_k(u\cdot v), \quad u, v \in S^{n-1},$$

we have

$$\| H_k(u\cdot) \|_{L^2(S^{n-1})} = \sqrt{H_k(1)}$$

Since

$$\begin{aligned} H_k(1) &= \frac{1}{\omega_n} \frac{2k+n-2}{n-2} P_k^{(n/2-1)}(1) \\ &= \frac{1}{\omega_n} \frac{(2k+n-2)(k+n-3)!}{(n-2)!\,k!} \\ &\leqslant \frac{2}{\omega_n} \left(k - 1 + \frac{n}{2} \right)^{n-2} \\ &\leqslant c_0 |\lambda_{n,k}^{n/2-1}|, \end{aligned}$$

where c_0 is a positive number independent of k. Substituting into the last inequality, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(u)| &\leqslant \sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^n|} \| H_k(u\cdot) \|_{L^2(S^{n-1})} \| \partial_v^{2n} f_k \|_{L^2(S^{n-1})} \\ &\leqslant \sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^{n/2+1}|} \| \partial_v^{2n} f_k \|_{L^2(S^{n-1})}\end{aligned}$$

$$\leqslant \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_{n,k}^{n/2+1}|} \right) \| \partial_v^{2n} f \|_{L^2(S^{n-1})}$$
$$\leqslant c \, \| \partial_v^{2n} f \|_{L^2(S^{n-1})}.$$

This proves the absolute and uniform convergence.

The fact that the series pointwisely converges to the function f itself is referred to [3].

Now, for $|x| < 1$, $x = \rho u$, $|u| = 1$, we have Poisson formula for equation (1)

$$U(x) = U(\rho u) = \int_{S^{n-1}} \mathscr{P}(\rho u, v) f(v) \, \mathrm{d}v$$
$$= \sum_{k=0}^{\infty} \rho^k \tau_k(\rho) \mathscr{H}_k f(u).$$

For $\rho \in [0, 1]$ we have $|\tau_k(\rho)| \leqslant 1$. This, together with the absolute and uniform convergence of the series $\sum f_k$, concludes that the series expanding $U(x)$ is absolutely and uniformly convergent in $\rho \in [0, 1)$ and $|u| = 1$. In a similar way we may show that the series of which the terms are the first- and second-order derivatives of those of the series defining $U(x)$ are also absolutely and uniformly convergent in the unit ball under the condition $f \in C^{4n}(S^{n-1})$. We therefore can apply the operator $\mathscr{L}$ to the series term by term, using its polar-co-ordinate form, and obtain

$$\mathscr{L} U(\rho u) = 0, \quad 0 \leqslant \rho < 1.$$

Owing to the uniform convergence for $\rho \in [0, 1]$ we can exchange the order of taking limit $\rho \to 1 - 0$ and the infinite summation, and, owing to $\lim_{\rho \to 1} \tau_k(\rho) = 1$, obtain

$$\lim_{\rho \to 1} U(\rho u) = f(u), \quad u \in S^{n-1}.$$

The uniform convergence of the series $\sum_{k=0}^{\infty} f_k$ guarantees the same uniform convergence of the above limit. The proof is complete. □

Remark. The proof of the theorem given above shows that the weaker condition $f \in C^{2([n/2]+3)}(S^{n-1})$ is enough to guarantee the conclusions of the theorem, where $[r]$ denotes the maximal integer that does not exceed r. A more precise result is proved in [4] using fractional powers of the operator $I - \partial_u^2$ and Sobolev space theory. For our purpose our theorem is sufficient, as in the sequel we shall use for $f \in C^{\infty}(S^{n-1})$.

3. Properties of the Poisson kernel

Theorem 1 implies the following boundary maximum principle.

Theorem 2. *If $f \in C^{4n}(S^{n-1})$ and U is the solution of* (1) *with the boundary value f in the sense of Theorem* 1, *then*

$$\min\{f(x) : |x| = 1\} \leqslant \min\{U(x) : |x| \leqslant 1\} \leqslant \max\{U(x) : |x| \leqslant 1\}$$
$$\leqslant \max\{f(x) : |x| = 1\}.$$

Proof. The boundary maximum principle of general elliptic equations implies that for any fixed $\rho < 1$ we have

$$\min\{U(x):|x| = \rho\} \leqslant \min\{U(x):|x| \leqslant \rho\} \leqslant \max\{U(x):|x| \leqslant \rho\}$$
$$\leqslant \max\{U(x):|x| = \rho\}.$$

Letting $\rho \to 1 - 0$, owing to the uniform convergence of Theorem 1, we conclude the desired inequalities. Theorem 2 is a particular case of a general result in [4].

The kernel $\mathscr{P}$ does not have an explicit expression in simple functions and so we do not have the convenience to see directly that the kernel has the δ-function properties. Nevertheless, using Theorems 1 and 2 we can still prove.

Theorem 3. *The kernel* $\mathscr{P}(\rho u, v)$, $0 \leqslant \rho < 1$, $|u| = |v| = 1$, *given by* (6) *has the following properties.*

(i) $\mathscr{P}(\rho u, v) > 0$;
(ii) $\forall v \in S^{n-1}, \frac{1}{\omega_n}\int_{S^{n-1}} \mathscr{P}(\rho u, v)\,\mathrm{d}v = 1$; *and*
(iii) $\forall \delta \in (0, 1]$, *for* $u \in S^{n-1}$ *uniformly* $\lim_{\rho\to 1}\int_{u\cdot v \leqslant 1-\delta} \mathscr{P}(\rho u, v)\,\mathrm{d}v = 0$.

Proof of (i). For $v \in S^{n-1}$ we shall use the notation $S(v, \imath)$ for the set $S^{n-1}\cap\{x \in \mathbb{R}^n : |x - v| < \imath\}$ on the sphere. Assume that $\mathscr{P}(\rho_0 u_0, v_0) < 0$ for some ρ_0, u_0, v_0. Since $\mathscr{P}$ is continuous in u and v, for the fixed u_0 we can find $\imath > 0$ such that $\mathscr{P}(\rho_0 u_0, v) < 0$ for all $v \in S(v_0, \imath)$. Let $f_{v_0,\imath}$ be a $C^\infty(S^{n-1})$-function satisfying $f_{v_0,\imath}(v) = 1$ for $v \in S(v_0, \imath/2)$, $f_{v_0,\imath}(v) = 0$ for $v \in S^{n-1}\backslash S(v_0, \imath)$ and $0 < f_{v_0,\imath}(v) < 1$ otherwise.

We then have

$$U(\rho_0 u_0) = \frac{1}{\omega_n}\int_{S^{n-1}} \mathscr{P}(\rho_0 u_0, v) f_{v_0,\imath}(v)\,\mathrm{d}v < 0.$$

On the other hand, Theorem 2 implies that $U(\rho u) \geqslant 0$ for all $\rho \in [0, 1)$ and $u \in S^{n-1}$. This is a contradiction. Property (i) is thus proved.

Proof of (ii). For $f \equiv 1$ on S^{n-1} using Theorems 1 and 2, we conclude (ii).

Proof of (iii). Let u_0 be fixed and $g_{u_0,\delta}$ be a $C^\infty(S^{n-1})$-function such that $g_{u_0,\delta}(u_0) = 0$, $g_{u_0,\delta}(v) = 1$ for those v such that $u_0\cdot v \leqslant 1 - \delta$ and $0 < g_{u_0,\delta}(v) < 1$ otherwise. Using properties (ii) and then (i), we have

$$U(\rho u_0) - g_{u_0,\delta}(u_0) = \frac{1}{\omega_n}\int_{S^{n-1}} \mathscr{P}(\rho u_0, v)(g_{u_0,\delta}(v) - g_{u_0,\delta}(u_0))\,\mathrm{d}v$$
$$\geqslant \frac{1}{\omega_n}\int_{u_0\cdot v \leqslant 1-\delta} \mathscr{P}(\rho u_0, v)\,\mathrm{d}v.$$

We therefore, conclude

$$\lim_{\rho \to 1-0} \int_{u_0 \cdot v < 1-\delta} \mathscr{P}(\rho u_0, v)\, \mathrm{d}v = 0.$$

Since $\mathscr{P}$ is rotationally invariant, we conclude the uniform convergence.

Having obtained Theorem 3, the proof of the following result is routine. For completeness, we still include its proof.

Theorem 4. *Let $f \in C(S^{n-1})$. Then the potential formula*

$$U(x) = \frac{1}{\omega_n} \int_{S^{n-1}} \mathscr{P}(x, v) f(v)\, \mathrm{d}v, \quad |x| < 1,$$

gives the solution to equation (1). *Moreover, if define $U(x) = f(x)$, $|x| = 1$, then the function U is continuous on the solid ball $\{|x| \leqslant 1\}$.*

Proof. In the case we have, for any $\delta > 0$,

$$\begin{aligned} U(\rho u) - f(u) &= \frac{1}{\omega_n} \int_{S^{n-1}} U(\rho u, v)(f(v) - f(u))\, \mathrm{d}v \\ &= \frac{1}{\omega_n} \int_{u \cdot v \leqslant 1-\delta} + \frac{1}{\omega_n} \int_{u \cdot v > 1-\delta} \\ &= I_1 + I_2. \end{aligned}$$

Let ε be given. We can choose δ such that $|f(u) - f(v)| < \varepsilon$ whenever $u \cdot v > 1 - \delta$. Then, owing to (i) and (ii) of Theorem 3, we have

$$|I_2| \leqslant \frac{\varepsilon}{\omega_n} \int_{u \cdot v > 1-\delta} \mathscr{P}(\rho u, v)\, \mathrm{d}v \leqslant \varepsilon$$

for any $\rho \in [0, 1)$. Now, owing to (iii) of Theorem 3, we can choose $\rho < 1$ close to 1 so that

$$\left| \frac{1}{\omega_n} \int_{u \cdot v \leqslant 1-\delta} \mathscr{P}(\rho u, v)\, \mathrm{d}v \right| \leqslant \frac{\varepsilon}{2 \| f \|_\infty}.$$

So, using (i),

$$\begin{aligned} |I_1| &\leqslant 2 \|f\|_\infty \frac{1}{\omega_n} \int_{u \cdot v \leqslant 1-\delta} \mathscr{P}(\rho u, v)\, \mathrm{d}v \\ &\leqslant \varepsilon. \end{aligned}$$

This proves (i). □

Since the convergence is uniform, the solution is continuous in the whole solid ball. Theorem 4 has the following:

Corollary. *Under the assumptions and notations in Theorem* 4, *defining $U_1(x) = U(x)$, $|x| < 1$; $U_1(x) = f(x)$, $|x| = 1$; and $U_1(x) = U(x/|x|^2)$, $|x| > 1$, we obtain that the* *function* U_1 *is continuous in* $\mathbb{R}^n$ *and satisfies* $\mathscr{L}U_1(x) = 0$, $|x| \neq 1$. *If in particular* f *is second-order differentiable, then* U_1 *is the solution of* $\mathscr{L}U_1(x) = 0$ *in the whole space* $\mathbb{R}^n$.

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