Adaptive rational approximation in Bergman space on bounded symmetric domain

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L. K. Hua in [3] constructed the Bergman kernel: for any $\quad, \quad \in \mathscr{D}$,

$$K(\ ,\ ) = \frac{1}{|\mathscr{D}|\det(I_q - \quad ^*\ )^{p+q}} \in A^2(\mathscr{D})$$

and established the reproducing property $\hat{j}(\ ) = \langle \hat{j}, K(\cdot,\ )\rangle$ for any $\hat{j} \in A^2(\mathscr{D})$.

In particular, $A^2(\mathscr{D})$ is a reproducing kernel Hilbert space with positive definite kernel function $\quad_w(\ ) = K(\ ,\ )$, and the linear span of all kernel functions $\{\ _w \mid\ \in \mathscr{D}\ \}$ is dense in $A^2(\mathscr{D})$. In the other words, one can use a finite linear combination of $\quad_{w_i}$ with $\quad_i \in \mathscr{D}$ to approximate any function in $A^2(\mathscr{D})$.

In the general setting of reproducing kernel Hilbert space (RKHS), the third author proposed POAFD scheme to approximate any element in underlying space with linear combination of kernel functions. We apply this proposal to Bergman space $A^2(\mathscr{D})$ setting in this paper, we outline the idea of POAFD below. More details can be found in our main Theorem 1.

Naively, for any given distinct points $\quad_1, \quad_2, \cdots, \quad_n$ of $\mathscr{D}$, the positive definite property of the reproducing kernel K of $A^2(\mathscr{D})$ implies that $\{\ _{a_1}, \quad_{a_2}, \cdots, \quad_{a_n}\}$ is linearly independent in $A^2(\mathscr{D})$, one then applies Gram-Schmidt orthonormalization to obtain an orthonormal sequence $\{\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n\}$ of $A^2(\mathscr{D})$ such that $\mathscr{B}_1 = \quad_{a_1}/\|\ _{a_1}\|$ and

$$\mathscr{B}_m = \frac{\quad_{a_m} - \sum_{i=1}^{m-1}\langle\ _{a_m}, \mathscr{B}_i\rangle\mathscr{B}_i}{\|\ _{a_m} - \sum_{i=1}^{m-1}\langle\ _{a_m}, \mathscr{B}_i\rangle\mathscr{B}_i\|} \tag{1}$$

for all $2 \le \quad \le \quad$. Then for any $\hat{j} \in A^2(\mathscr{D})$, let $\quad_n$ be the image of the orthogonal projection of $\hat{j}$ onto the span$\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ = span$\{\ _{a_i} \mid\ = 1, \cdots,\ \}$,

$$\quad_n = \sum_{i=1}^n \langle \hat{j}, \mathscr{B}_i\rangle\mathscr{B}_i \quad\text{and}\quad \|\ _n\|^2 = \sum_{i=1}^n |\langle \hat{j}, \mathscr{B}_i\rangle|^2. \tag{2}$$

In [4,8,5], T. Qian proposed the *maximal selection principle* to select an optimal sequence $\{\ _1, \cdots, \ _n\}$ of points in $\mathscr{D}$ successively such that the modulus $|\langle \hat{j}, \mathscr{B}_i\rangle|$ of the coefficients in (2) is as large as possible for $\quad = 1, 2, \cdots,\ $.

In particular, when $\quad = 1$, the first point $\quad_1 \in \mathscr{D}$ is chosen to satisfy the following

$$\left|\left\langle \hat{j}, \frac{\quad_{a_1}}{\|\ _{a_1}\|}\right\rangle\right| = \sup\left\{\ \left|\left\langle \hat{j}, \frac{\quad_b}{\|\ _b\|}\right\rangle\right| \ \middle|\ \blacktriangleright \in \mathscr{D}\ \right\}. \tag{3}$$

One can establish the existence of $\quad_1$ by means of boundary vanishing property (BVP) in Proposition 4 below. For the other points $\quad_m$ ($2 \le \quad \le \quad$) in $\mathscr{D}$, we establish their existence inductively by means of studying the following objective function $\quad_m$ defined by

$$\quad_m(\blacktriangleright) = |\langle\ \hat{j},\ \mathscr{B}_m^b\ \rangle|, \tag{4}$$

for all $\blacktriangleright \in \mathscr{D} \setminus \{\ _1, \cdots,\ _{m-1}\}$, where $\mathscr{B}_m^b = \frac{k_b - \sum_{i=1}^{m-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i}{\|k_b - \sum_{i=1}^{m-1}\langle k_b, \mathscr{B}_i\rangle\mathscr{B}_i\|}$.

With the help of BVP again, one can prove that $\quad_m$ can be extended to a bounded continuous function on $\overline{\mathscr{D}}_m = \overline{\mathscr{D}} \setminus \{\ _1, \cdots,\ _{m-1}\}$. Then the sup$\{\ (\blacktriangleright) \mid \blacktriangleright \in \overline{\mathscr{D}}_m\ \}$ can be realized by a limit $\lim_{i\to\infty}\ _m(\blacktriangleright_i)$ where $\{\blacktriangleright_i\}_{i\ge 1}$ is a sequence in $\overline{\mathscr{D}}_m$. By passing to its subsequence if necessary, we may assume that the sequence $\{\blacktriangleright_i\}_{i\ge 1}$ converges to a point $\quad$ in $\overline{\mathscr{D}}$. One can use BVP to rule out the case that $\quad$ lies in the boundary of $\mathscr{D}$, so $\quad \in \mathscr{D}$. Then we have the following two cases.

In the case of $\quad \notin \{\ _1, \cdots,\ _{m-1}\}$, we set $\quad_m = \quad$ and $\mathscr{B}_m$ as in (1). However, the other case $\quad \in \{\ _1, \cdots,\ _{m-1}\}$ poses a serious problem, one can still set $\quad_m = \quad$, which implies that the $\quad$ points

$_1$, $_2$,···, $_m$ are not distinct, so the set $\{_{a_1},\cdots,_{a_m}\}$ is linearly dependent. For this reason, we follow the idea of [8,10] by introducing the following generalized kernels

$$_{a,\alpha}(\ ) = \frac{|\alpha|}{-\alpha}K(\ ,\ )\Big|_{w=a}$$

for any multi-index $\ \in \mathbb{N}^{pq}$.

In this case, we set $_m = $. By analyzing the function $_m$ in more detail, we will prove in section 4 that there exist a positive integer ℓ and a sequence $(_\alpha)_{|\alpha|=\ell}$ of complex numbers with all $\ \in \mathbb{N}^{pq}$ of order $|\ | = \ell$, such that

$$E = \sum_{|\alpha|=\ell} \alpha\ _{a,\alpha} \neq \mathbf{0}, \quad \text{and} \quad \mathscr{B}_m = \frac{E - \sum_{i=1}^{m-1}\langle E, \mathscr{B}_i\rangle\mathscr{B}_i}{\|E - \sum_{i=1}^{m-1}\langle E, \mathscr{B}_i\rangle\mathscr{B}_i\|} \in A^2(\mathscr{D}).$$

In both cases, the following maximal modulus property holds:

$$|\langle \hat{\jmath}, \mathscr{B}_m\rangle| = \sup\left\{\ \left|\left\langle \hat{\jmath}, \frac{_b - \sum_{i=1}^{m-1}\langle\ _b, \mathscr{B}_i\rangle\mathscr{B}_i}{\|\ _b - \sum_{i=1}^{m-1}\langle\ _b, \mathscr{B}_i\rangle\mathscr{B}_i\|}\right\rangle\right|\ \Big|\ \blacktriangleright \in \mathscr{D}\setminus\{\ _1,\cdots,\ _{m-1}\}\ \right\}. \tag{5}$$

We can summarize all of these in the following.

Ⲧ... 1. (Maximal selection principle). *Let $\mathscr{D}$ be a bounded symmetric domain of type I. For any $\hat{\jmath} \in A^2(\mathscr{D})$, there exist*

  (i) *a sequence $(_i)_{i\geq 1}$ of points in $\mathscr{D}$,*
  (ii) *a sequence $(_{a_1,m_1},\ _{a_2,m_2},\cdots,\ _{a_i,m_i},\cdots)$ of generalized kernel functions in $A^2(\mathscr{D})$, where $\ _i \in \mathbb{N}$ and*

$$_{a_i,m_i}(\ ) = \sum_{|\alpha|=m_i} _{i,\alpha}\frac{|\alpha|\ K}{-\alpha}(\ ,\ )\Big|_{w=a_i}$$

  for some $\ _{i,\alpha} \in \mathbb{C}$,
  (iii) *an orthonormal sequence $\{\mathscr{B}_1,\cdots,\mathscr{B}_i,\cdots\}$ of rational functions in $A^2(\mathscr{D})$ obtained by applying Gram-Schmidt orthonormalization process to the sequence in (ii), such that the following maximal selection principle holds:*

$$|\langle \hat{\jmath}, \mathscr{B}_i\rangle| = \sup\left\{\ |\langle \hat{\jmath}, \mathscr{B}_i^b\rangle|\ \Big|\ \blacktriangleright \in \mathscr{D}\setminus\{\ _1,\ _2,\cdots,\ _{i-1}\}\ \right\} \tag{6}$$

  for all $\ \geq 1$.

It follows from our main Theorem 1 that for any $\hat{\jmath} \in A^2(\mathscr{D})$, we can construct an orthonormal sequence $\{\mathscr{B}_i\}_{i\geq 1}$.

Ⲧ... 2. (Convergence of POAFD). *With the notations stated above, the sequence $\sum_{i=1}^n\langle\hat{\jmath}, \mathscr{B}_i\rangle\mathscr{B}_i$ in $A^2(\mathscr{D})$ converges to $\hat{\jmath}$ in $A^2(\mathscr{D})$, i.e.,*

$$\hat{\jmath} = \sum_{i\geq 1}\langle\hat{\jmath}, \mathscr{B}_i\rangle\mathscr{B}_i \qquad \text{in} \quad A^2(\mathscr{D}). \tag{7}$$

The right hand side of equation (7) is called a *pre-orthogonal adaptive Fourier decomposition* (POAFD) of $\hat{j}$. Similar results could be found in [1], and the monograph [6] written in Chinese is still one of the best source of the recent development of adaptive Fourier decomposition (AFD), including some applications of AFD and its variants in signal processing.

The proof of Theorem 1 is quite long as it occupies most of the paper. The rest of this paper is structured as follows. In section 2, we review some basic facts of the irreducible bounded symmetric domain $\mathscr{D}$ of Type I, the Bergman space $A^2(\mathscr{D})$ and the Bergman kernel K. In section 3, we introduce generalized kernel functions $_{a,\alpha}$ and proved that they form a linearly independent set. We show that $A^2(\mathscr{D})$ satisfies boundary vanishing property which is related to the maximal selection principle in Theorem 1. In section 4, we establish a maximal selection principle in $A^2(\mathscr{D})$ by finding an explicit limit $\mathscr{B}_i$ in $A^2(\mathscr{D})$ so that the optimal value of $|\langle \hat{j}, \mathscr{B}_i^b \rangle|$ on $\mathscr{D}$ can be attained. In section 5, we give a proof of Theorem 2.

2. ⟍ ﹍ s ﹍ $A^2(\mathscr{D})$

2.1. Notations

Let $\mathbb{N}$ be the set of all non-negative integers. Denote by $\mathbb{N}^{pq}$ the set of all multi-indices $= (_{1,1}, \ldots, _{p,q})$ of nonnegative integers. We use $\geq '$ to mean $_{i,j} \geq '_{i,j}$ for all $= 1, \ldots,$ and $= 1, \ldots$. In particular, ≥ 0 means all $_i$'s are nonnegative. We also write $| | = _{1,1} + \cdots + _{p,q}, ! = _{1,1}! \cdots _{p,q}!$.

For any fixed positive integers $\geq$, let $\mathbb{C}^{p \times q}$ be the set of all complex $\times$ matrices $(_{ij})_{1 \leq i,j \leq q}$.

A Hermitian matrix $A = (A_{ij})_{1 \leq i,j \leq q} \in \mathbb{C}^{q \times q}$ is called *positive semi-definite*, denoted by $A \geq 0$, if

$$\mathbf{z}^* A \mathbf{z} = \sum_{1 \leq i,j \leq q} \overline{_i} A_{ij} _j \geq 0 \tag{8}$$

for any column vector $\mathbf{z} = (_1, _2, \cdots, _q)^T \in \mathbb{C}^q$. Furthermore, A is called *positive definite*, denoted by $A > 0$, when the following condition holds:

$$\mathbf{z}^* A \mathbf{z} = 0 \text{ holds if and only if } \mathbf{z} = \mathbf{0} \text{ in } \mathbb{C}^q.$$

Let I_q be the $\times$ identity matrix. Let $\mathscr{D} = \{ \in \mathbb{C}^{p \times q} \mid I_q - ^* > 0 \}$ be an irreducible bounded symmetric domain $\mathscr{D}$ of Type I in the Helgason book [2]. This is a non-empty connected open subset in $\mathbb{C}^{p \times q}$. The condition $I_q - ^* > 0$ implies that $\mathscr{D}$ is bounded in $\mathbb{C}^{p \times q}$, so it has a finite Euclidean volume, denoted by $|\mathscr{D}|$. Our basic reference is Hua's classical treatise [3]. In the following, we will use I to denote I_q. Let $\mathcal{O}(\mathscr{D})$ be the vector space of the holomorphic functions defined on the domain $\mathscr{D}$. Obviously, the coordinate functions $_{i,j}( ) = _{i,j}$ are holomorphic on $\mathscr{D}$.

We use $\mathbb{C}^{pq}$ to denote the usual -dimensional complex vector space. From now on, we use $\mathbb{C}^{pq}$ instead of $\mathbb{C}^{p \times q}$, and equip $\mathbb{C}^{pq}$ with standard Euclidean norm $\| \cdot \|$. A power series in $\mathbb{C}^{pq}$ with center $\in \mathbb{C}^{pq}$ is of the form

$$\sum_{\alpha \geq 0} {}_\alpha(-)^\alpha = \sum_{\alpha_{1,1} \geq 0, \ldots, \alpha_{p,q} \geq 0} {}_{\alpha_{1,1} \ldots \alpha_{p,q}} (_{1,1} - _{1,1})^{\alpha_{1,1}} \cdots (_{p,q} - _{p,q})^{\alpha_{p,q}}, \tag{9}$$

where $_{\alpha_{1,1} \ldots \alpha_{p,q}}$'s are complex numbers and

$$(-)^\alpha = (_{1,1} - _{1,1})^{\alpha_{1,1}} \cdots (_{p,q} - _{p,q})^{\alpha_{p,q}}.$$

For holomorphic and anti-holomorphic derivatives of functions on domain in $\mathbb{C}^{pq}$, we write

$$\frac{^{|\alpha|}}{\partial \alpha} = \frac{^{|\alpha|}}{_{1,1}^{\alpha_{1,1}} \cdots _{p,q}^{\alpha_{p,q}}} \quad \text{and} \quad \frac{^{|\beta|}}{\partial \overline{\beta}} = \frac{^{|\beta|}}{\overline{}_{1,1}^{\beta_{1,1}} \cdots \overline{}_{p,q}^{\beta_{p,q}}},$$

$$\hat{j}^{(\alpha)}(\ ) = \frac{\partial^{|\alpha|}\hat{j}}{\partial\ ^\alpha}(\ )\quad\text{and}\quad \hat{j}^{(\overline{\beta})}(\ ) = \frac{\partial^{|\beta|}\hat{j}}{\partial\ _{-\beta}}(\ ),$$

in which we use a bar on top of the order multi-index   for the anti-holomorphic derivatives of order   .
Similarly, if $\hat{j}(\ ,\ )$ is defined on some open subset of $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$, we denote

$$\hat{j}^{(\alpha,\overline{\beta})}(\ ,\ ) = \frac{\partial^{|\alpha|+|\beta|}\hat{j}}{\partial\ ^\alpha\,\partial\ _{-\beta}}(\ ,\ ).$$

2.2. Reproducing kernel of $A^2(\mathscr{D})$

We first recall the definition of reproducing kernel Hilbert space [11].

1. Let $\mathcal{H}$ be a Hilbert space of complex-valued functions defined on a non-empty set   with an inner product $\langle\ ,\ \rangle$. $\mathcal{H}$ is called a *reproducing kernel Hilbert space* (RKHS) on   , if for any point $\in$ the evaluation functional $L_x : \mathcal{H} \to \mathbb{C}$ defined by $L_x(\hat{j}) = \hat{j}(\ )$ is continuous on $\mathcal{H}$.

The following proposition follows from the Riesz representation theorem.

s 1. *Let $\mathcal{H}$ be a RKHS on   . Then for any $\in$, there exists a unique function ${}_y \in \mathcal{H}$ such that $\hat{j}(\ ) = \langle \hat{j},\ _y \rangle$ for all $\hat{j} \in \mathcal{H}$.*

2. Let $\mathcal{H}$ be a RKHS on   and $\in$. The function ${}_y$ in Proposition 1 is called the *reproducing kernel function* at the point $\in$. In this case, the function $K :\ \times\ \to \mathbb{C}$ defined by

$$K(\ ,\ ) =\ _y(\ )$$

is called the *reproducing kernel* for $\mathcal{H}$.

1. *The linear span of the set $\{\ _y \mid\ \in\ \}$ of all reproducing kernel functions ${}_y$ is dense in $\mathcal{H}$.*

 . Let V be the closure of the linear span of $\{\ _y \mid\ \in \mathcal{H}\}$ in $\mathcal{H}$. Suppose contrary that V is a proper closed subspace in $\mathcal{H}$. Then there exists a non-zero $\hat{j}$ in the orthogonal complement $V^\perp$ of V in $\mathcal{H}$. $\hat{j}$ is orthogonal to ${}_y$ for all $\in$ and thus $\hat{j}(\ ) = \langle \hat{j},\ _y \rangle = 0$ for every $\in$, i.e., $\hat{j} = \mathbf{0}$ in $\mathcal{H}$, which is a contradiction. Hence, $V = \mathcal{H}$. □

3. *[3] (i) The Bergman space $A^2(\mathscr{D})$ is a reproducing kernel Hilbert space and the reproducing kernel is the Bergman kernel in the form*

$$K(\ ,\ ) = |\mathscr{D}|^{-1} \cdot (\det(I -\ ^*\ ))^{-(p+q)}. \tag{10}$$

(ii) The reproducing kernel functions ${}_w(\ ) = K(\ ,\ )$ satisfy the following reproducing property: for any $\in \mathscr{D}$ and for any $\hat{j} \in A^2(\mathscr{D})$,

$$\langle \hat{j},\ _w \rangle = \hat{j}(\ ).$$

(iii) The linear span of the set $\{\ _w \mid\ \in \mathscr{D}\}$ is dense in $A^2(\mathscr{D})$, i.e., any function in $A^2(\mathscr{D})$ can be approximated by linear combination of ${}_w$'s.

The domain $\mathscr{D}$ in $\mathbb{C}^{pq}$ can be viewed as generalizations of the unit disc $\mathbb{D}$ in $\mathbb{C}$ when $q = 1$, and of the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$ when $q = 1$ respectively.

The following example shows that the classical Bergman space $A^2(\mathbb{B}^p)$ on the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$ is a special case of the space $A^2(\mathscr{D})$. The corresponding topological boundaries and the Bergman kernels are recorded.

1. For $q = 1$, we have

$$\mathscr{D} = \left\{ \mathbf{z} = (z_1, \ldots, z_p) \in \mathbb{C}^p \mid \mathbf{z}^*\mathbf{z} = \sum_{k=1}^{p} |z_k|^2 < 1 \right\},$$

which is the unit ball $\mathbb{B}^p$ in $\mathbb{C}^p$, and

$$\mathbb{B}^p = \left\{ \mathbf{z} \in \mathbb{C}^p \mid \mathbf{z}^*\mathbf{z} = 1 \right\}$$

is its boundary. The Bergman kernel for the Bergman space $A^2(\mathbb{B}^p)$ is given by the following rational function in $\mathbf{z}$ and $\overline{\mathbf{w}}$:

$$K(\mathbf{z}, \mathbf{w}) = \frac{1}{|\mathbb{B}^p|(1 - \mathbf{w}^*\mathbf{z})^{p+1}}. \quad \square$$

The topological boundary $\partial \mathscr{D}$ of $\mathscr{D}$ is the disjoint union of q connected components $\mathcal{B}_1, \cdots, \mathcal{B}_q$, where

$$\mathcal{B}_i = \left\{ \Lambda \in \mathbb{C}^{pq} \mid I - \Lambda^*\Lambda \geq 0 \text{ and has rank } q - i \right\}$$

for $1 \leq i \leq q$, and $\mathcal{B}_{i+1} \subset \overline{\mathcal{B}}_i$ for $1 \leq i \leq q - 1$.

The following lemma will be used in the proof of the boundary vanishing property (BVP) of $A^2(\mathscr{D})$ in Proposition 4.

2. *Let $A \in \mathscr{D}$ and $B \in \overline{\mathscr{D}} = \mathscr{D} \cup \partial \mathscr{D}$. Then,*

(i) $\det(I - A^*B) \neq 0$;
(ii) *if $A_0 \in \partial \mathscr{D}$, then $\lim\limits_{A \to A_0} \det(I - A^*A) = 0$;*
(iii) *the function $\hat{j}_A(B) = |\det(I - A^*B)|^{-1}$ extends continuously to $\overline{\mathscr{D}}$ and $\hat{j}_A(B)$ is bounded for any $B \in \overline{\mathscr{D}}$.*

(i) Suppose contrary that $\det(I - A^*B) = 0$, as $I - A^*B$ is a $q \times q$ matrix, there exists a non-zero $\mathbf{v} \in \mathbb{C}^q$ such that $(I - A^*B)\mathbf{v} = \mathbf{0}$. Then we have $0 = \mathbf{v}^*\mathbf{0} = \mathbf{v}^*(I - A^*B)\mathbf{v} = \mathbf{v}^*\mathbf{v} - \mathbf{v}^*A^*B\mathbf{v} = \|\mathbf{v}\|^2 - (A\mathbf{v})^*(B\mathbf{v})$, which implies that

$$\|\mathbf{v}\|^2 = \langle B\mathbf{v}, A\mathbf{v} \rangle. \tag{11}$$

On the other hand, as $A \in \mathscr{D}$, $B \in \overline{\mathscr{D}}$ and $\mathbf{v} \neq \mathbf{0} \in \mathbb{C}^q$, we have

$$\mathbf{v}^*(I - A^*A)\mathbf{v} > 0 \quad \text{and} \quad \mathbf{v}^*(I - B^*B)\mathbf{v} \geq 0,$$

and hence we have $\|\mathbf{v}\|^2 = \mathbf{v}^*\mathbf{v} = \mathbf{v}^*I\mathbf{v} > \mathbf{v}^*A^*A\mathbf{v} = (A\mathbf{v})^*(A\mathbf{v}) = \|A\mathbf{v}\|^2$, and $\|\mathbf{v}\|^2 \geq \|B\mathbf{v}\|^2$ similarly. Then $\|A\mathbf{v}\|\|B\mathbf{v}\| \leq \|A\mathbf{v}\|\|\mathbf{v}\| < \|\mathbf{v}\|^2$. It follows from (11) and Cauchy-Schwarz inequality that

$$\|\mathbf{v}\|^2 = \langle B\mathbf{v}, A\mathbf{v} \rangle = |\langle B\mathbf{v}, A\mathbf{v} \rangle| \leq \|B\mathbf{v}\|\|A\mathbf{v}\| < \|\mathbf{v}\|^2,$$

which is a contradiction.

(ii) As $\varphi(A) = \det(I - A^*A)$ is a polynomial in the entries of A and their conjugates, it is continuous on $\overline{\mathscr{D}}$. For any point A_0 in the boundary $\partial\mathscr{D}$, the rank of $I - A_0^*A_0$ is less than q, $\lim_{A \to A_0} \varphi(A) = \varphi(A_0) = 0$.

(iii) For a fixed $A \in \mathscr{D}$, by (i), the function $\psi(B) = (\det(I - A^*B))^{-1}$ is bounded on $\mathscr{D}$, and hence the result follows. $\square$

3. $\quad$ s $\quad A^2(\mathscr{D})$

In the one-dimensional case [10], the authors constructed generalized kernel functions by differentiating the reproducing kernel functions iteratively. In this section, we generalize their idea to high-dimensional case.

We are going to define the generalized kernel functions $\kappa_{a,\alpha}$ from a given kernel $\kappa_a(\cdot) = K(\cdot, \cdot)$ by taking anti-holomorphic derivatives of order α with respect to second variable, and prove some important and useful properties which will be used in later sections.

3.1. Generalized kernel functions

3. For any given $a \in \mathscr{D}$, $\alpha \in \mathbb{N}^{pq}$, define *generalized kernel function* $\kappa_{a,\alpha}$ on $\mathscr{D}$ of order α at the point a to be

$$\kappa_{a,\alpha}(\cdot) = \left.\frac{\partial^{|\alpha|}}{\partial \overline{w}^\alpha} \kappa_w\right|_{w=a} = \left.\frac{\partial^{|\alpha|}}{\partial \overline{w}^\alpha} K(\cdot, \cdot)\right|_{w=a}. \tag{12}$$

In particular, $\kappa_{a,\alpha} = \kappa_a$ if $\alpha = (0, \ldots, 0) \in \mathbb{N}^{pq}$.

By Theorem 3(iii), any function $\hat{j} \in A^2(\mathscr{D})$ can be approximated by the linear combination of kernel functions κ_a on $\mathscr{D}$. For any given $\hat{j} \in A^2(\mathscr{D})$, we want to select a sequence $\{a_i\}_{i \geq 1}$ of points in $\mathscr{D}$ successively and to apply Gram-Schmidt orthonormalization process to construct a sequence $\{\mathscr{B}_i\}_{i \geq 1}$ of orthonormal rational functions in $A^2(\mathscr{D})$.

We first study some properties of kernel functions κ_a and their anti-holomorphic derivatives. By

(i) $\hat{j}(\ ,\ ) = \overline{\hat{j}(\ ,\ )}$ for all $(\ ,\ ) \in \mathbb{C}^{pq} \times \mathbb{C}^{pq}$;

(ii) $\hat{j}(\ ,\ )$ is holomorphic in the first variable $\ $;

(iii) $\hat{j}(\ ,\ )$ is anti-holomorphic in the second variable $\ $.

Then for any $\ \in \mathbb{N}^{pq}$, we have $\hat{j}^{(\alpha,\overline{\beta})}(\blacktriangleright,\ ) = \overline{\hat{j}^{(\beta,\overline{\alpha})}(\ ,\blacktriangleright)}$.

$\diagdown$ s   **2.** (i) $\ _{a,\alpha} \in A^2(\mathscr{D})$ for any $\ \in \mathscr{D}$ and $\ \in \mathbb{N}$

$$= \sum_{\mathbf{a}\in P} \sum_{k=0}^{m} \left(\sum_{|\alpha|=k} \overline{}_{\mathbf{a},\alpha}^{-\alpha} \right)^{k} \exp(\langle \mathbf{a}, \ \rangle).$$

As $P \subset \mathscr{D} \subset \mathbb{C}^{pq}$ and there are only finitely many vectors in P, there exists a non-empty open subset U in $\mathbb{C}^{pq}$ such that

(U$_1$) $^{\alpha} \neq 0$ for all of order at most for all $\in U$, and

(U$_2$) for any $\in U$, the complex numbers $\langle \mathbf{a}, \ \rangle$ are distinct for $\mathbf{a} \in P$.

It follows from condition (U$_2$) that for any fixed $\in U$, the set of the following functions in :
$\{ \ ^{k} \exp(\langle \mathbf{a}, \ \rangle) \mid \mathbf{a} \in P, \ 0 \leq \ \leq \ \}$ is linearly independent, which implies that the following complex coefficients vanish for any fixed $\in U$:

$$\sum_{|\alpha|=k} \overline{}_{\mathbf{a},\alpha} \ ^{\alpha} = \sum_{|\alpha|=k} \overline{}_{\mathbf{a},\alpha}^{-\alpha} = 0.$$

This vanishing condition of the homogeneous polynomial in of degree at most on non-empty open subset U on $\mathbb{C}^{pq}$ and condition (U$_1$) implies that all the coefficients $_{\mathbf{a},\alpha}$ in the polynomial above are all zero as well. $\square$

4. (Cauchy estimate) *Let $\hat{j}$ be a holomorphic function on a -dimensional polydisc centered at $\in \mathbb{C}^n$ with positive radius R', and*

$$\hat{j}(\ ) = \sum_{\alpha \geq 0} \frac{\hat{j}^{\alpha}(\ )}{!} \ ^{\alpha}$$

be the Taylor series of $\hat{j}$ centered at $=$. Suppose that $R < R'$ and let

$$M_R(\hat{j}) = \max \left\{ \ |\hat{j}(\ )| \ \middle| \ = (\ _1, \cdots, \ _n) \in \ , \ |\ _i - \ _i| \leq R \right\}.$$

Then for all $\in \mathbb{N}^n$, we have $\left| \dfrac{\hat{j}^{(\alpha)}(\ )}{!} \right| \leq \dfrac{M_R(\hat{j})}{R^{|\alpha|}}.$ $\square$

Next we will prove that the Taylor series of the Bergman kernel in the anti-holomorphic variables converges in $A^2(\mathscr{D})$.

3. *Let $\in \mathscr{D} \subset \mathbb{C}^{pq}$, $\mathbf{v} \in \mathbb{C}^{pq}$ and $\in \mathbb{C}$ with sufficiently small $|\ |$ such that $+ \ \mathbf{v} \in \mathscr{D}$, then*

(i) *the series $\sum_{\alpha \geq 0} \ _{a,\alpha} \frac{(\overline{t\mathbf{v}})^{\alpha}}{\alpha!}$ converges in $A^2(\mathscr{D})$; and*

(ii) *in the norm topology of $A^2(\mathscr{D})$, we have*

$$_{a+t\mathbf{v}} = \sum_{\alpha \geq 0} \frac{\ _{a,\alpha}}{!} (\overline{\ \mathbf{v}})^{\alpha}.$$

(i) As $K(\ , \ ) = \frac{1}{|\mathscr{D}| \det(1-\overline{w}^T z)^{p+q}}$ and it is holomorphic in $\in \mathbb{C}^{pq}$ and is anti-holomorphic in $\in \mathbb{C}^{pq}$, or equivalently it is holomorphic in $\overline{} \in \mathbb{C}^{pq}$. For any multi-indices $, \ \in \mathbb{N}^{pq}$, and $, \blacktriangleright \in \mathscr{D}$, it follows from Cauchy estimate in Lemma 4 that

$$\frac{|K^{(\beta,\overline{\alpha})}(\blacktriangleright, \ )|}{!\ !} \leq \frac{M_R(K)}{R^{|\alpha|+|\beta|}}, \tag{18}$$

where $M_R(K) = \max\{\, |K(\,,\,)| \mid |\,_{ij} - \blacktriangleright_{ij}| \leq R, \text{ and } |\,_{ij} - \,_{ij}| \leq R \text{ for all }\,,\,\}$. Moreover, by Proposition 2(ii), $\langle \,_{a,\alpha},\,_{b,\beta}\rangle = K^{(\beta,\overline{\alpha})}(\blacktriangleright,\,)$. Suppose $\in \mathbb{C}$ with sufficiently small $|\,|$, and any $\mathbf{v} \in \mathbb{C}^{pq}$, it follows from (16) that $\|\,_{a,\alpha}\|^2 = \langle \,_{a,\alpha},\,_{a,\alpha}\rangle = K^{(\alpha,\overline{\alpha})}(\,,\,)$, and then from (18) and geometric series that

$$
\sum_{\alpha \geq 0}\left\|\,_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{!}\right\| \leq \sum_{\alpha \geq 0}\|\,_{a,\alpha}\|\frac{|(\,\mathbf{v})^\alpha|}{!} \leq \sum_{\alpha \geq 0}\sqrt{\frac{M_R(K)}{R^{2|\alpha|}}}|(\,\mathbf{v})^\alpha|
$$

$$
\leq \sqrt{M_R(K)}\sum_{\alpha \geq 0}\left|\left(\frac{\mathbf{v}}{R}\right)^\alpha\right| = \frac{\sqrt{M_R(K)}}{\prod_{ij}(1 - \frac{t|v_{ij}|}{R})}. \tag{19}
$$

This implies that the series $\sum_{\alpha \geq 0}\,_{a,\alpha}\frac{(\overline{t\mathbf{v}})^\alpha}{\alpha!}$ converges in $A^2(\mathscr{D})$.

(ii) By expanding the function $K(\,+\,\mathbf{v},\blacktriangleright+\,\mathbf{w})$ into power series of the holomorphic parameters $\,\mathbf{v}$ and anti-holomorphic parameter $\overline{\,\mathbf{w}}$, we have

$$
\|\,_{a+t\mathbf{v}}\|^2 = K(\,+\,\mathbf{v},\,+\,\mathbf{v}) = \sum_{\alpha,\beta \geq 0}\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta, \tag{20}
$$

in which the series converges uniformly for all $\mathbf{v}$ in the unit ball of $\mathbb{C}^{pq}$ as long as $|\,|$ is sufficiently small. Then by Proposition 2(ii) and using the Taylor series expansion in (20), the result follows from the following estimate

$$
\left\|\,_{a+t\mathbf{v}} - \sum_{|\beta| \leq N}\,_{a,\beta}\frac{(\overline{\,\mathbf{v}})^\beta}{!}\right\|^2
$$

$$
\leq \|\,_{a+t\mathbf{v}}\|^2 - 2\sum_{|\beta| \leq N}\mathrm{Re}\langle \,_{a,\beta}\frac{(\overline{\,\mathbf{v}})^\beta}{!},\,_{a+t\mathbf{v}}\rangle + \sum_{|\alpha|,|\beta| \leq N}\mathrm{Re}\langle \,_{a,\beta}\frac{(\overline{\,\mathbf{v}})^\beta}{!},\,_{a,\alpha}\frac{(\overline{\,\mathbf{v}})^\alpha}{!}\rangle
$$

$$
= K(\,+\,\mathbf{v},\,+\,\mathbf{v}) - 2\sum_{|\beta| \leq N}\mathrm{Re}\left(K^{(\mathbf{0},\overline{\beta})}(\,+\,\mathbf{v},\,)\frac{(\overline{\,\mathbf{v}})^\beta}{!}\right)
$$

$$
+ \sum_{|\alpha|,|\beta| \leq N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta\right)
$$

$$
= \sum_{\alpha,\beta \geq 0}\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta - 2\sum_{\alpha \geq 0,|\beta| \leq N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta\right)
$$

$$
+ \sum_{|\alpha|,|\beta| \leq N}\mathrm{Re}\left(\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta\right)
$$

$$
= \mathrm{Re}\left(\sum_{|\alpha| \geq N,|\beta| \geq N}\frac{K^{(\alpha,\overline{\beta})}(\,,\,)}{!\,!}(\,\mathbf{v})^\alpha(\overline{\,\mathbf{v}})^\beta\right).
$$

The last sum is the tail of the Taylor series of $K(\,+\,\mathbf{v},\,+\,\mathbf{v})$ in which converges uniformly to 0 for all unit vectors $\mathbf{v}$ of $\mathbb{C}^{pq}$ as $N \to +\infty$, provided that $|\,|$ is sufficiently small. $\square$

3.2. Boundary vanishing property

In order to prove our main result the maximum selection principle in Theorem 1, it is crucial to check that the reproducing kernel function $\,_a$ has the boundary vanishing property in Proposition 4(i). We start with the following easy lemma and proposition.

Lemma 5. *For any $\hat{j} \in A^2(\mathscr{D})$ and any $a \in \mathscr{D}$, define $H(a) = \varphi_a \in A^2(\mathscr{D})$ and $G(a) = |\langle \hat{j}, \varphi_a \rangle|$, where $\varphi_a = \frac{k_a}{\|k_a\|}$. Then H and G are continuous on $\mathscr{D}$.*

Proof. As $\|k_a\| > 0$ for all $a \in \mathscr{D}$, it suffices to show that the function $H(a) = \varphi_a$ is continuous from $\mathscr{D}$ to $A^2(\mathscr{D})$. In fact, as $K(\cdot, \cdot)$ is continuous on $\mathscr{D} \times \mathscr{D}$,

$$\|\varphi_b - \varphi_a\|^2 = \|\varphi_b\|^2 + \|\varphi_a\|^2 - 2\mathrm{Re}\langle \varphi_b, \varphi_a \rangle = K(b,b) + K(a,a) - 2\mathrm{Re}K(b,a)$$

converges to 0 as b approaches a. □

Next, we will prove that the function G in Lemma 5 can be extended continuously to $\overline{\mathscr{D}}$.

Proposition 4. *(i) (**Boundary vanishing property**) For any $\hat{j} \in A^2(\mathscr{D})$ and any $b_0 \in \partial\mathscr{D}$, we have $\lim_{b \to b_0} |\langle \hat{j}, \varphi_b \rangle| = 0$.*
(ii) For any $\hat{j} \in A^2(\mathscr{D})$, there exists $a \in \mathscr{D}$ such that $|\langle \hat{j}, \varphi_a \rangle| = \max\{|\langle \hat{j}, \varphi_b \rangle| \mid b \in \mathscr{D}\}$.

Proof. (i) By Lemma 1, the linear span of $\{\varphi_a \mid a \in \mathscr{D}\}$ is a dense subspace of $A^2(\mathscr{D})$. For any $\epsilon > 0$, there exist a positive integer N, finite sequences $\{a_i\}_{i=1}^N \subset \mathscr{D}$ and $\{c_i\}_{i=1}^N \subset \mathbb{C}$ such that $F = \sum_{i=1}^N c_i \varphi_{a_i}$, and $\|\hat{j} - F\| < \epsilon$. Since $\|\varphi_b\| = 1$, the triangle inequality and the Cauchy-Schwarz inequality give

$$|\langle \hat{j}, \varphi_b \rangle| - |\langle F, \varphi_b \rangle| \le |\langle \hat{j} - F, \varphi_b \rangle| \le \|\hat{j} - F\| \, \|\varphi_b\| < \epsilon.$$

For any $b \in \mathscr{D}$, by (15), we have

$$|\langle \hat{j}, \varphi_b \rangle| \le |\langle F, \varphi_b \rangle| + \epsilon = \sqrt{|\mathscr{D}|} \, |\det(I - b^*b)|^{\frac{p+q}{2}} \, |F(b)| + \epsilon$$
$$\le |\mathscr{D}|^{-\frac{1}{2}} \, |\det(I - b^*b)|^{\frac{p+q}{2}} \sum_{i=1}^N |c_i| \, |\det(I - a_i^*b)|^{-(p+q)} + \epsilon. \tag{21}$$

By Lemma 2(ii) and 2(iii), the function $\psi(b) = |\det(I - a_i^*b)|^{-(p+q)}$ is bounded on $\overline{\mathscr{D}}$ for all $a_i \in \mathscr{D}$. Therefore, the N-sum $\sum_{i=1}^N |c_i| \, |\det(I - a_i^*b)|^{-(p+q)}$ in (21) is bounded. Since $\lim_{b \to b_0} \det(I - b^*b) = 0$, the result follows.
(ii) By Lemma 5, $G(b) = |\langle \hat{j}, \varphi_b \rangle|$ is continuous on $\mathscr{D}$. By (i), $G(b)$ can be continuously extended to the compact subset $\overline{\mathscr{D}}$ in $\mathbb{C}^{pq}$. Then, consider the following two cases of $M = \max\{G(b) \mid b \in \overline{\mathscr{D}}\}$.

If $M = 0$, then $0 \le |\langle \hat{j}, \varphi_b \rangle| \le M = 0$ for all $b \in \mathscr{D}$. By reproducing property of φ_b, $\hat{j} = 0$ on $\mathscr{D}$. In this case, choose a to be any point in $\mathscr{D}$.

If $M > 0$, then it follows from BVP that $G(b) = 0$ for all $b \in \partial\mathscr{D}$, i.e., the maximum M can only be attained at some interior point of $\overline{\mathscr{D}}$. Therefore, there exists $a \in \mathscr{D}$ such that $M = G(a) = |\langle \hat{j}, \varphi_a \rangle|$. □

4. Main result

The goal of this section is to prove Theorem 1 about the existence of pre-orthogonal adaptive Fourier decomposition in $A^2(\mathscr{D})$. We will establish our main result Theorem 1 by a series of lemmas and propositions in the rest of this section. We first construct the sequence $(a_n)_{n \ge 1}$ of points in $\mathscr{D}$ needed in Theorem 1(i) inductively on n.

4.1. Initial step for n_1

Proof. For $n = 1$, define $\hat{j}_1 = \hat{j}$. The BVP in Proposition 4 implies that there exists $a_1 \in \mathscr{D}$ such that

$$\left| \left\langle \hat{j}_1, \frac{a_1}{\|a_1\|} \right\rangle \right| = \sup \left\{ \left| \left\langle \hat{j}_1, \frac{b}{\|b\|} \right\rangle \right| \; \middle| \; b \in \mathscr{D} \right\},$$

which is the equality (6) of Theorem 1. In this initial step, we set

$$_1 = 0, \quad _{a_1,m_1} = \; _{a_1} \quad \text{and} \quad \mathscr{B}_1 = \; _{a_1} = \frac{_{a_1}}{\| \; _{a_1} \|}.$$

Then in the inductive step, let's first assume that we have a sequence $(\;_1, \cdots, \;_n)$ of not necessarily distinct points in $\mathscr{D}$, and a sequence $(\;_{a_1,m_1}, \;_{a_2,m_2}, \cdots, \;_{a_n,m_n})$ in $A^2(\mathscr{D})$ as stated in Theorem 1. By applying Gram-Schmidt orthonormalization process to $(\;_{a_i,m_i})_{1 \le i \le n}$, we obtain an orthonormal set $\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ in $A^2(\mathscr{D})$. Define $\hat{j}_j \in A^2(\mathscr{D})$ be the -th residual function as follows:

$$\hat{j}_1 = \hat{j}, \quad \text{and} \quad \hat{j}_j = \hat{j}_{j-1} - \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle \mathscr{B}_{j-1} \tag{22}$$

for $2 \le \; \le \; + 1$. We can deduce the following.

5. *Let $\hat{j} \in A^2(\mathscr{D})$, $\{\mathscr{B}_1, \cdots, \mathscr{B}_n\}$ be an orthonormal set in $A^2(\mathscr{D})$ and $\{\hat{j}_j\}_{1 \le j \le n+1}$ be defined in (22), then we have*

(i) $\langle \hat{j}_j, \mathscr{B}_{j-1} \rangle = \langle \hat{j}_j, \mathscr{B}_{j-2} \rangle = \cdots = \langle \hat{j}_j, \mathscr{B}_1 \rangle = 0$;

(ii) $\langle \hat{j}_j, \mathscr{B}_j \rangle = \langle \hat{j}_{j-1}, \mathscr{B}_j \rangle = \cdots = \langle \hat{j}_2, \mathscr{B}_j \rangle = \langle \hat{j}_1, \mathscr{B}_j \rangle = \langle \hat{j}, \mathscr{B}_j \rangle$ *for* $1 \le \; \le \;$;

(iii) $\hat{j}_j = \hat{j}_\ell - \sum_{i=\ell}^{j-1} \langle \hat{j}, \mathscr{B}_i \rangle \mathscr{B}_i = \hat{j} - \sum_{i=1}^{j-1} \langle \hat{j}, \mathscr{B}_i \rangle \mathscr{B}_i$ *for* $1 \le \ell < \; \le \; + 1$;

(iv) $\text{span}\{ \mathscr{B}_1, \cdots, \mathscr{B}_{j-1}, \mathscr{B}_j \} = \text{span}\{ \;_{a_1,m_1}, \;_{a_2,m_2}, \cdots, \;_{a_j,m_j} \}$ *for* $1 \le \; \le \;$;

(v) $\langle \hat{j}_j, \;_{a_i,m_i} \rangle = 0$ *for* $1 \le \; < \; \le \; + 1$;

(vi) $_b - \sum_{j=1}^n \langle \;_b, \mathscr{B}_j \rangle \mathscr{B}_j = \mathbf{0}$ *if and only if* $\blacktriangleright \in \{ \;_1, \cdots, \;_n \}$.

(i) Using $\hat{j}_j = \hat{j}_{j-1} - \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle \mathscr{B}_{j-1}$ and $\langle \mathscr{B}_i, \mathscr{B}_j \rangle = \;_{ij}$, one has

$$\langle \hat{j}_j, \mathscr{B}_{j-1} \rangle = \langle \hat{j}_{j-1} - \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle \mathscr{B}_{j-1}, \mathscr{B}_{j-1} \rangle$$
$$= \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle - \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle \| \mathscr{B}_{j-1} \|^2 = 0,$$

the others follow similarly.

(ii) follows from $\langle \hat{j}_{i+1}, \mathscr{B}_j \rangle = \langle \hat{j}_i - \langle \hat{j}_i, \mathscr{B}_i \rangle \mathscr{B}_i, \mathscr{B}_j \rangle = \langle \hat{j}_i, \mathscr{B}_j \rangle$ if $\; < \;$.

(iii) follows from (ii) and $\hat{j}_j = \hat{j}_{j-1} - \langle \hat{j}_{j-1}, \mathscr{B}_{j-1} \rangle \mathscr{B}_{j-1} = \hat{j}_{j-1} - \langle \hat{j}, \mathscr{B}_{j-1} \rangle \mathscr{B}_{j-1}$.

(iv) The result follows from Gram-Schmidt orthonormalization process.

(v) The result follows from (iv).

(vi) If $\blacktriangleright \notin \{ \;_1, \cdots, \;_n \}$, then by Proposition 2(iii), the set $\{ \;_{a_1,m_1}, \cdots, \;_{a_n,m_n} \}$ is linearly independent. If $\blacktriangleright \in \{ \;_1, \cdots, \;_n \}$, then $_b \in \text{span}\{ \;_{a_1,m_1}, \cdots, \;_{a_n,m_n} \}$. Hence, by (iv), we have

$$_b - \sum_{j=1}^n \langle \;_b, \mathscr{B}_j \rangle \mathscr{B}_j = \mathbf{0} \iff \;_b \in \text{span}\{ \mathscr{B}_1, \cdots, \mathscr{B}_n \}$$

$$\iff \;_b \in \text{span}\{ \;_{a_1,m_1}, \cdots, \;_{a_n,m_n} \} \iff \blacktriangleright \in \{ \;_1, \cdots, \;_n \}. \quad \square$$

The following lemma is an application of Theorem 1(iii), and it justifies for the existence in the search of $\;_{n+1}$.

6. *Suppose that $\hat{j}_j \neq 0$ $(1 \le \; \le \;)$, then $|\langle \hat{j}_j, \mathscr{B}_j \rangle| \neq 0$.*

$`$ , $`$. As $\hat{j}_j \neq 0$, it follows from continuity of $\hat{j}_j$ that there exist $\in \mathscr{D}\backslash\{ _1, \cdots, _{j-1}\}$ and > 0 such that $|\hat{j}_j( )| > 0$ for all point in the open ball B centered at with radius . As $\{ _1, \cdots, _{j-1}\}$ is a finite set in $\mathscr{D}$, one can find a point $\in B \setminus \{ _1, \cdots, _{j-1} \} \subseteq \mathscr{D} \setminus \{ _1, \cdots, _{j-1} \}$.

As $\neq _i$ $(1 \leq \leq -1)$, Proposition 5(vi) implies that $_z, \mathscr{B}_1, \cdots, \mathscr{B}_{j-1}$ are linearly independent, and so

$$0 < \|\ _z - \sum_{i=1}^{j-1} \langle\ _z, \mathscr{B}_i \rangle \mathscr{B}_i \| - 1 \langle\ \mathcal{B} \cdot \text{B} \rightarrow\} \ \rightarrow\!\!\rightarrow\!\!\rangle \quad \rightarrow\!\!\rightarrow\!\!\rightarrow \mathcal{B} \ \rightarrow\} \ \rightarrow\!\!\rightarrow \quad \rightarrow \quad \rightarrow \ \rightarrow\!\!\rightarrow \quad \rightarrow\!\!\!\rightarrow \|$$

4. Let $\mathscr{D}_n = \mathscr{D} \setminus \{\ _1, \cdots, \ _n\}$ be the punctured domain. Define an objective function $\quad$ on $\mathscr{D}_n$ as follows: $(\blacktriangleright) = |\langle \hat{j}_{n+1}, \mathscr{B}_{n+1}^b \rangle|$ for any $\blacktriangleright \in \mathscr{D}_n$.

It follows from Proposition 5(i) and Cauchy-Schwarz inequality that $\langle \hat{j}_{n+1}, \mathscr{B}_j \rangle = 0$,

$$
(\blacktriangleright) = |\langle \hat{j}_{n+1}, \mathscr{B}_{n+1}^b \rangle| = \left| \left\langle \hat{j}_{n+1}, \frac{b - \sum_{j=1}^n \langle\ _b, \mathscr{B}_j \rangle \mathscr{B}_j}{\| \ _b - \sum_{j=1}^n \langle\ _b, \mathscr{B}_j \rangle \mathscr{B}_j \|} \right\rangle \right|
$$

$$
= \frac{|\langle \hat{j}_{n+1}, \ _b \rangle|}{\| \ _b - \sum_{j=1}^n \langle\ _b, \mathscr{B}_j \rangle \mathscr{B}_j \|} = \frac{|\langle \hat{j}_{n+1}, \dot{\bullet}_b \rangle|}{\| \dot{\bullet}_b - \sum_{j=1}^n \langle \dot{\bullet}_b, \mathscr{B}_j \rangle \mathscr{B}_j \|}
$$

$$
= \frac{|\langle \hat{j}_{n+1}, \dot{\bullet}_b \rangle|}{\sqrt{1 - \sum_{j=1}^n 255}}
$$

In this case, the equality (6) of Theorem 1 obviously holds.

We will consider the case ( ) in the next subsection.

4.4. The case ($\mathbf{B}$) of repeatedly selected points

In the following, we assume that the limit point appeared in the sequence ($_1, \cdots, _n$) of points in $\mathscr{D}$. As $\blacktriangleright_m \in \mathscr{D}_n$, so $\blacktriangleright_m \neq$, one can define a sequence $\{\mathbf{v}_m\}_{m \geq 1}$ of unit vectors in $\mathbb{C}^{pq}$ and a sequence $\{_m\}_{m \geq 1}$ of positive numbers as follows: for any $\in \mathbb{Z}_+$,

$$_m = \|\blacktriangleright_m - \| > 0,$$
$$\mathbf{v}_m = \frac{\blacktriangleright_m - }{\|\blacktriangleright_m - \|} = \frac{\blacktriangleright_m - }{_m} \in \mathbb{C}^{pq}.$$

In particular, $\blacktriangleright_m = + _m \mathbf{v}_m$. Then $= \lim\limits_{m \to \infty} \blacktriangleright_m$ implies that $\lim\limits_{m \to \infty} {}_m = 0$.

As the sequence $\{\mathbf{v}_m\}_{m \geq 1}$ is a sequence in the compact unit sphere in $\mathbb{C}^{pq}$, it has a convergent subsequence, still denoted by $\{\mathbf{v}_m\}_{m \geq 1}$, with limit $\mathbf{v} = (\mathbf{v}(1), \mathbf{v}(2), \cdots, \mathbf{v}(\text{?}))$ in the unit sphere of $\mathbb{C}^{pq}$.

Define $_m = + _m \mathbf{v}$ for all . Since $\lim\limits_{m \to \infty} {}_m = 0$ and is an interior point of $\mathscr{D}$, we may assume that $_m \in \mathscr{D}$ for all ≥ 1. Moreover, $\lim\limits_{m \to \infty} {}_m = = \lim\limits_{m \to \infty} \blacktriangleright_m$. As Recall that the supremum S in (26) is defined by

$$S = \lim_{m \to \infty} (\blacktriangleright_m) = \lim_{m \to \infty} |\langle \tilde{j}_{n+1}, \mathscr{B}_{n+1}^{b_m} \rangle|. \tag{27r66 3j /3 3}$$

As the limit point appears in the sequence $(\lambda_i)_{1\leq i\leq n}$ in the case (ι), choose the smallest index such that $\lambda = \lambda_i$, then $c_i = 0$. By Proposition 5(i), $\langle \hat{\jmath}_{n+1}, \mathscr{B}_j \rangle = 0$ for $1 \leq j \leq$, and Proposition 5(iv) implies that $0 = \langle \hat{\jmath}_{n+1}, \psi_{a_i,m_i} \rangle = \langle \hat{\jmath}_{n+1}, \psi_{a_i,0} \rangle = \hat{\jmath}_{n+1}(\lambda_i) = \hat{\jmath}_{n+1}(\lambda)$. The facts that $\hat{\jmath}_{n+1} \neq 0$ and $\hat{\jmath}_{n+1}(\lambda) = 0$ imply that $\hat{\jmath}$ is not a constant function. Hence, there exists $\in \mathbb{N}^{pq}$ such that $\hat{\jmath}_{n+1}^{(\alpha)}(\lambda) \neq 0$.

. There exists $\ell \geq 1$ such that

$$\sum_{|\alpha|=\ell} \hat{\jmath}_{n+1}^{(\alpha)}(\lambda)\frac{\mathbf{v}^\alpha}{\alpha!} \neq 0.$$

. Suppose contrary that for any $\ell \geq 1$,

$$\sum_{|\alpha|=\ell} \hat{\jmath}_{n+1}^{(\alpha)}(\lambda)\frac{\mathbf{v}^\alpha}{\alpha!} = 0,$$

which implies that for sufficiently large ,

$$0 = \sum_{\alpha\geq 0} \hat{\jmath}_{n+1}^{(\alpha)}(\lambda)\frac{\mathbf{v}^\alpha}{\alpha!}\gamma_m^\alpha = \hat{\jmath}_{n+1}(\lambda + \gamma_m\mathbf{v}) = \hat{\jmath}_{n+1}(\lambda_m).$$

It follows from the equality in (25) that $\hat{\jmath}(\lambda_m) = 0$, and hence $S = \lim_{m\to\infty} \hat{\jmath}(\lambda_m) = 0$, which contradicts to $\hat{\jmath}_{n+1} \neq 0$. □

5. Let ℓ be the least positive integer such that

(i) $\displaystyle\sum_{|\alpha|=j} \hat{\jmath}_{n+1}^{(\alpha)}(\lambda)\frac{\mathbf{v}^\alpha}{\alpha!} = 0$ for all $j = 0, 1, \cdots, \ell - 1$;

(ii) $\displaystyle\sum_{|\alpha|=\ell} \hat{\jmath}_{n+1}^{(\alpha)}(\lambda)\frac{\mathbf{v}^\alpha}{\alpha!} \neq 0$.

With the help of definition, we define $E = \displaystyle\sum_{|\alpha|=\ell} \psi_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$ which is nonzero in $A^2(\mathscr{D})$.

6. For the case (ι) of the limit point stated above, we define

(i) $\lambda_{n+1} = \lambda$, $j_{n+1} = \ell$,

(ii) $\psi_{a_{n+1},m_{n+1}} = \psi_{a,\ell} = \displaystyle\sum_{|\alpha|=\ell} \psi_{a,\alpha}\frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$, and

(iii) $\mathscr{B}_{n+1} = \dfrac{\psi_{a_{n+1},m_{n+1}} - \sum_{i=1}^n \langle \psi_{a_{n+1},m_{n+1}}, \mathscr{B}_i \rangle \mathscr{B}_i}{\| \psi_{a_{n+1},m_{n+1}} - \sum_{i=1}^n \langle \psi_{a_{n+1},m_{n+1}}, \mathscr{B}_i \rangle \mathscr{B}_i \|}$.

In the following, we will prove that Theorem 1 holds for selection of the triple $(\lambda_{n+1}, j_{n+1}, \mathscr{B}_{n+1})$ stated above.

4.5. Asymptotic analysis

The main result of this subsection is the following Theorem 4. This shows that in the case (ι), where the limit point of the sequence $\{\lambda_m = \lambda + \gamma_m\mathbf{v}\}_{m\geq 1}$ appears in the sequence $(\lambda_1, \ldots, \lambda_n)$, we can always

construct a function $\mathscr{B}_{n+1}$ in $A^2(\mathscr{D})$ associated to   such that $\{\mathscr{B}_1, \ldots, \mathscr{B}_n, \mathscr{B}_{n+1}\}$ is an orthonormal set. We finally can state the most important result in this paper, but its proof requires some delicate estimates.

4. *The supremum S in* (26) *is given by*

$$S = \lim_{m \to \infty} \ (\blacktriangleright_m) = \left| \left\langle \hat{\jmath}_{n+1}, \frac{E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i}{\| E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i \|} \right\rangle \right|.$$

We divide the proof of the Theorem 4 into a series of lemmas below.

     . Let   , $\in \mathbb{N}^{pq}$ and $\ell \in \mathbb{N}$. Define

(i) $S_\ell = \{ \quad \in \mathbb{N}^{pq} \mid \ | \ | \ge \ell \ \}$.
(ii) $T_\alpha = \{ \quad \in \mathbb{N}^{pq} \mid \quad \ge \quad \} = \{ \quad + \quad \in \mathbb{N}^{pq} \mid \quad \ge 0 \ \}$.

     . $S_\ell = \bigcup_{|\alpha|=\ell} T_\alpha$.   □

     . *In the setup above, we have*

$$\lim_{m \to \infty} \frac{\hat{\jmath}_{n+1}(\blacktriangleright_m)}{\frac{\ell}{m}} = \sum_{|\alpha|=\ell} \frac{\hat{\jmath}_{n+1}^{(\alpha)}(\ )}{!} \mathbf{v}^\alpha.$$

     . As stated in Remark 1, we assume that the sequence $\{\blacktriangleright_m\}$ in $\mathscr{D}$ is defined by $\blacktriangleright_m = \quad + \quad_m \mathbf{v}$ for all ≥ 1. Then we write down the Taylor series of $\hat{\jmath}_{n+1}$ centered at   . It follows from Definition 5 on the least positive integer ℓ and for sufficiently large   that we have

$$\hat{\jmath}_{n+1}(\blacktriangleright_m) = \hat{\jmath}_{n+1}(\ + \ _m \mathbf{v}) = \ _m^\ell \left(\sum_{|\alpha|=\ell} \hat{\jmath}_{n+1}^{(\alpha)}(\ ) \frac{\mathbf{v}^\alpha}{!} + \sum_{|\alpha|>\ell} \hat{\jmath}_{n+1}^{(\alpha)}(\ ) \frac{\mathbf{v}^\alpha}{!} \ _m^{|\alpha|-\ell} \right).$$

Now we want to estimate the second sum, by means of Lemma 8 and Cauchy estimate in Lemma 4. For sufficiently large   , we have

$$\left| \sum_{|\alpha|>\ell} \frac{\hat{\jmath}_{n+1}^{(\alpha)}(\ )}{!} \ _m^{|\alpha|} \mathbf{v}^\alpha \right| \le \sum_{|\alpha|>\ell} \left| \frac{\hat{\jmath}_{n+1}^{(\alpha)}(\ )}{!} \right| \ _m^{|\alpha|} |\mathbf{v}^\alpha|$$

$$= \sum$$

$$\leq \frac{\ell+1}{m} \sum_{|\beta|=\ell+1} |\mathbf{v}^\beta| \, M_{2t_m}(\hat{j}_{n+1}^{(\beta)}) \sum_{\delta\geq 0} \left|\left(\frac{\mathbf{v}}{2}\right)^\delta\right|$$

$$= \frac{\ell+1}{m} \sum_{|\beta|=\ell+1} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(\hat{j}_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(\ )/2|)}, \tag{28}$$

where $M_{2t_m}(\hat{j}_{n+1}^{(\beta)}) = \max\left\{ |\hat{j}_{n+1}^{(\beta)}(\ +\mathbf{r})| \ \Big| \ \|\mathbf{r}\|_\infty \leq 2_m \right\}$, and
$\mathbf{v} = (\mathbf{v}(1), \mathbf{v}(2), \cdots, \mathbf{v}(^q))$.

The summands in (28) satisfy

$$\lim_{m\to\infty} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(\hat{j}_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(\ )/2|)} = \frac{|\mathbf{v}^\beta| \, |\hat{j}_{n+1}^{(\beta)}(\ )|}{\prod_{i=1}^{pq}(1-|\mathbf{v}(\ )/2|)} < \infty,$$

so we have

$$\lim_{m\to\infty} \frac{1}{\frac{\ell}{m}} \left| \sum_{|\alpha|>\ell} \frac{\hat{j}_{n+1}^{(\alpha)}(\ )}{!} {}^{|\alpha|}_m \mathbf{v}^\alpha \right| \leq \lim_{m\to\infty} {}_m \sum_{|\beta|=\ell+1} \frac{|\mathbf{v}^\beta| \, M_{2t_m}(\hat{j}_{n+1}^{(\beta)})}{\prod_{i=1}^{pq}(1-|\mathbf{v}(\ )/2|)} = 0,$$

and hence $\displaystyle\lim_{m\to\infty} \frac{\hat{j}_{n+1}(\boldsymbol{\flat}_m)}{\frac{\ell}{m}} = \sum_{|\alpha|=\ell} \frac{\hat{j}_{n+1}^{(\alpha)}(\ )}{!} \mathbf{v}^\alpha.$ □

As $S = \displaystyle\lim_{m\to\infty}(\boldsymbol{\flat}_m) > 0$ in (26), and

$$S = \lim_{m\to\infty} \left(\frac{\frac{|\langle f_{n+1}, k_{b_m}\rangle|}{t_m^\ell}}{\frac{\|k_{b_m}-\sum_{i=1}^n \langle k_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{t_m^\ell}} \right), \tag{29}$$

we have

$$\lim_{m\to\infty} \frac{\| {}_{b_m} - \sum_{i=1}^n \langle {}_{b_m}, \mathscr{B}_i\rangle \mathscr{B}_i\|}{\frac{\ell}{m}} = \frac{\displaystyle\lim_{m\to\infty} \frac{|\langle f_{n+1}, k_{b_m}\rangle|}{t_m^\ell}}{S}, \tag{30}$$

which is finite and non-zero.

. In the notations above, we define

$$T_m = \sum_{|\alpha|<\ell} {}_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{!} {}^{|\alpha|}_m \quad \text{and} \quad {}_m = {}_{b_m} - T_m - {}^\ell_m E,$$

where $E = \sum_{|\alpha|=\ell} {}_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!}$ is defined in Theorem 4.

It is obvious that $T_m, E, {}_m$ are in $A^2(\mathscr{D})$. And it follows from $\boldsymbol{\flat}_m = \ + {}_m\mathbf{v}$ and Proposition 3 that in the norm of $A^2(\mathscr{D})$ we have

$$T_m + {}^\ell_m E + {}_m = {}_{b_m} = {}_{a+t_m\mathbf{v}} = \sum_{\alpha\geq 0} {}_{a,\alpha} \frac{\overline{\mathbf{v}}^\alpha}{!} {}^{|\alpha|}_m, \tag{31}$$

which implies that the following holds in the norm of $A^2(\mathscr{D})$,

$$\Phi_m = \sum_{|\alpha|>\ell} a_{\bar{a},\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} \Phi_m^{|\alpha|}. \tag{32}$$

Then it follows from (31) that

$$
\Psi_{b_m} - \sum_{i=1}^{n} \langle \Psi_{b_m}, \mathscr{B}_i \rangle \mathscr{B}_i = \left(T_m - \sum_{i=1}^{n} \langle T_m, \mathscr{B}_i \rangle \mathscr{B}_i \right)
$$
$$
+ \left(E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i \right) \Phi_m^\ell + \left(\Phi_m - \sum_{i=1}^{n} \langle \Phi_m, \mathscr{B}_i \rangle \mathscr{B}_i \right). \tag{33}
$$

10. $\displaystyle\lim_{m\to\infty} \frac{\| \Phi_m - \sum_{i=1}^{n} \langle \Phi_m, \mathscr{B}_i \rangle \mathscr{B}_i \|}{\Phi_m^\ell} = 0.$

As $\{\mathscr{B}_1, \ldots, \mathscr{B}_n\}$ is an orthonormal set in $A^2(\mathscr{D})$, $\sum_{i=1}^{n} \langle \Phi_m, \mathscr{B}_i \rangle \mathscr{B}_i$ is the orthogonal projection of Φ_m onto the subspace spanned by $\{\mathscr{B}_1, \ldots, \mathscr{B}_n\}$, so $\| \Phi_m - \sum_{i=1}^{n} \langle \Phi_m, \mathscr{B}_i \rangle \mathscr{B}_i \| \le \| \Phi_m \|$. It suffices to show the limit $\displaystyle\lim_{m\to\infty} \frac{\| \Phi_m \|}{\Phi_m^\ell} = 0$. For any $\phi \in A^2(\mathscr{D})$ and for sufficiently large m, it follows from (32) that

$$
\langle \phi, \Phi_m \rangle = \langle \phi, \sum_{|\alpha|>\ell} a_{\bar{a},\alpha} \frac{\overline{\mathbf{v}}^\alpha}{\alpha!} \Phi_m^{|\alpha|} \rangle = \sum_{|\alpha|>\ell} \frac{\phi^{(\alpha)}(\ )}{\alpha!} \mathbf{v}^\alpha \Phi_m^{|\alpha|}.
$$

Following the same idea of the proof of Lemma 9, we have

$$
|\langle \phi, \Phi_m \rangle| \le \sum_{|\alpha|\ge\ell+1} \left| \frac{\phi^{(\alpha)}(\ )}{\alpha!} \right| |\mathbf{v}|^\alpha \Phi_m^{|\alpha|} \le \Phi_m^{\ell+1} \sum_{|\beta|=\ell+1} |\mathbf{v}^\beta| \frac{M_{2t_m}(\phi^{(\beta)})}{\prod_{i=1}^{pq}(1 - \frac{|\mathbf{v}(i)|}{2})}. \tag{34}
$$

In particular, we can estimate $\frac{\| \Phi_m \|^2}{\Phi_m^{2\ell}} = \frac{\langle \Phi_m, \Phi_m \rangle}{\Phi_m^{2\ell}}$ by replacing $\phi = \Phi_m$ in (34).

For this, we need to show the following estimate

$$
M_{2t_m}(\Phi_m^{(\beta)}) = \max \left\{ |\Phi_m^{(\beta)}(\ + 2 t_m \mathbf{r})| \ \Big| \ \|\mathbf{r}\|_\infty \le 1 \right\}
$$
$$
\le \Phi_m^{\ell+1} \max_{\|\mathbf{r}\|_\infty \le 1} \left\{ \sum_{|\gamma|=\ell+1} |\mathbf{v}^\gamma| \frac{M_{2t_m}(K_{a+2t_m\mathbf{r}}^{(\gamma,\overline{\beta})})}{} \right.
$$

$$= \left| \sum_{|\alpha|>\ell} \frac{\overline{K^{(\alpha,\overline{\beta})}(\ ,\ +2\ _m\mathbf{r})}}{!} \overline{\mathbf{v}}^\alpha\ _m^{|\alpha|} \right| \leq \sum_{|\alpha|>\ell} \left| \frac{K^{(\alpha,\overline{\beta})}(\ ,\ +2\ _m\mathbf{r})}{!} \right| |\mathbf{v}^\alpha|\ _m^{|\alpha|}$$

$$\leq\ _m^{\ell+1} \sum_{|\gamma|=\ell+1} |\mathbf{v}^\gamma| \frac{M_{2t_m}(K^{(\gamma,\overline{\beta})}_{a+2t_m\mathbf{r}})}{\prod_{i=1}^{pq}\left(1-\frac{|\mathbf{v}(i)|}{2}\right)}.$$

In the last inequality, we apply the same technique in the proof of the estimate (28) in Lemma 9. Moreover, one can easily use the continuity of the function $K^{(\gamma,\overline{\beta})}(\ ,\ +2\ _m\mathbf{r})$ to show that the finite sum of the last expression has a finite limit as   tends to infinity.

The final result follows from the extra factor $\ _m^2$ in (34) and (35) as follows:

$$\frac{\|\ _m\|^2}{\ _m^{2\ell}} \leq\ _m^2 \sum_{|\beta|=|\gamma|=\ell+1} \frac{|\mathbf{v}^\beta||\mathbf{v}^\gamma| \max_{\|\mathbf{r}\|_\infty \leq 1}\left\{M_{2t_m}(K^{(\gamma,\overline{\beta})}_{a+2t_m\mathbf{r}})\right\}}{\prod_{i=1}^{pq}\left(1-\frac{|\mathbf{v}(i)|}{2}\right)^2},$$

in which the last finite double sum converges to finite number as   converges to infinity. □

11. *Let* $E_j = \sum_{|\alpha|=j} \frac{\overline{\mathbf{v}}^\alpha}{!}\ _{a,\alpha}$ *and* $F_j = E_j - \sum_{i=1}^n \langle E_j, \mathscr{B}_i \rangle \mathscr{B}_i$. *Let* $J = \{\ \ |\ 1 \leq\ \ \leq \ell-1,\ F_j \neq 0\ \}$. *Then* $\{F_j\}_{j\in J}$ *is linearly independent.*

Suppose that $\sum_{j\in J}\ _j F_j = 0$, we want to prove that $\ _j = 0$ for all $\ \in J$ as follows. Then the assumption implies that

$$\sum_{j\in J}\ _j E_j = \sum_{j\in J}\sum_{i=1}^n\ _j \langle E_j, \mathscr{B}_i\rangle \mathscr{B}_i. \tag{36}$$

It follows from Proposition 5(iv) $\mathrm{span}\{\mathscr{B}_1,\cdots,\mathscr{B}_n\} = \mathrm{span}\{\ _{a_1,m_1},\cdots,\ _{a_n,m_n}\}$ that there exist $A_1,\cdots,A_n \in \mathbb{C}$ such that

$$\sum_{j\in J}\ _j E_j = \sum_{i=1}^n A_i\ _{a_i,m_i}. \tag{37}$$

Let $I = \{\ \ |\ 1 \leq\ \ \leq\ ,\ _i =\ \ \}$. By definition, each E_j is a linear combination of generalized kernel functions of same total order   at the same point   , it follows from (37), Proposition 2(iii) and Proposition 5(iv) that for each $\ \in J$, there exist $C_1,\cdots,C_n \in \mathbb{C}$ such that

$$\ _j E_j = \sum_{i\in I, m_i=j} A_i\ _{a,m_i} = \sum_{i=1}^n C_i \mathscr{B}_i,$$

$$\ _j F_j =\ _j E_j - \sum_{i=1}^n \langle\ _j E_j, \mathscr{B}_i\rangle \mathscr{B}_i = \mathbf{0}.$$

As $\ \in J$, so $F_j \neq \mathbf{0}$, and hence $\ _j = 0$ for all $\ \in J$. □

Finally, Lemma 10 and Lemma 11 imply that the following.

12. $T_m - \sum_{i=1}^n \langle T_m, \mathscr{B}_i\rangle \mathscr{B}_i = \mathbf{0}$ *in* $A^2(\mathscr{D})$ *for any* ≥ 1.

`、` `、`. Apply the triangle inequality to (33), we have

$$
\frac{\| \, b_m - \sum_{i=1}^{n} \langle \, b_m , \mathscr{B}_i \rangle \mathscr{B}_i \|}{\frac{\ell}{m}} + \| \quad m
$$

$$= \lim_{m \to \infty} \left\| \left(E + \frac{\frac{m}{\ell}}{m} \right) - \sum_{k=1}^{n} \left\langle \left(E + \frac{\frac{m}{\ell}}{m} \right), \mathscr{B}_k \right\rangle \mathscr{B}_k \right\|$$

$$= \left\| E - \sum_{k=1}^{n} \left\langle E, \mathscr{B}_k \right\rangle \mathscr{B}_k \right\|.$$

Suppose contrary that the limit above is zero, then $E = \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i$ which is non-zero vector in $A^2(\mathscr{D})$. Proposition 5(i) implies that $\langle \hat{j}_{n+1}, \mathscr{B}_i \rangle = 0$ for all $= 1, \cdots,$, and hence

$$\langle \hat{j}_{n+1}, E \rangle = \sum_{i=1}^{n} \overline{\langle E, \mathscr{B}_i \rangle} \langle \hat{j}_{n+1}, \mathscr{B}_i \rangle = 0,$$

which violates the choice of E made in Definition 5(ii). $\square$

4. By Lemma 9 and Lemma 13, we have

$$S = \lim_{m \to \infty} (\blacktriangleright_m) = \lim_{m \to \infty} |\langle \hat{j}_{n+1}, \mathscr{B}_{n+1}^{b_m} \rangle| = \lim_{m \to \infty} \left(\frac{\frac{|\langle f_{n+1}, k_{b_m} \rangle|}{t_m^{\ell}}}{\frac{\|k_{b_m} - \sum_{i=1}^{n} \langle k_{b_m}, \mathscr{B}_i \rangle \mathscr{B}_i \|}{t_m^{\ell}}} \right)$$

$$= \frac{\left| \sum_{|\alpha|=\ell} \frac{f_{n+1}^{(\alpha)}(a)}{\alpha!} \mathbf{v}^{\overline{\alpha}} \right|}{\left\| \sum_{|\alpha|=\ell} \frac{\overline{\mathbf{v}}^{\alpha}}{\alpha!} \left({}_{a,\alpha} - \sum_{i=1}^{n} \langle {}_{a,\alpha}, \mathscr{B}_i \rangle \mathscr{B}_i \right) \right\|}$$

$$= \left| \left\langle \hat{j}_{n+1}, \frac{E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i}{\|E - \sum_{i=1}^{n} \langle E, \mathscr{B}_i \rangle \mathscr{B}_i \|} \right\rangle \right|, \tag{40}$$

which completes the proof of Theorem 4. $\square$

Note that $\mathscr{B}_{n+1}$ in Definition 6(iii) agrees with (40). This completes the inductive step in constructing the sequences (i) and (ii) in Theorem 1. However, it still remains to check the maximal modulus property (MSP) in Theorem 1(iii) in order to complete the proof of Theorem 1 as follows.

1. We have defined $\mathscr{B}_1$ in the initial step and $\mathscr{B}_{n+1}$ in the inductive step for cases ( ) $\in \mathscr{D}_n$ and ( ) $\in \{ {}_1, \cdots, {}_n \}$ respectively, such that the equality (40) implies

$$|\langle \hat{j}_{n+1}, \mathscr{B}_{n+1} \rangle| = \sup \left\{ |\langle \hat{j}_{n+1}, \mathscr{B}_{n+1}^b \rangle| \; \middle| \; \blacktriangleright \in \mathscr{D} \setminus \{ {}_1, {}_2, \cdots, {}_n \} \right\} = S, \tag{41}$$

where $\mathscr{B}_{n+1}$ is given in Definition 6(iii) as in (40). Since $\mathscr{B}_i$ ($= 1, \cdots,$) are mutually orthonormal, by Proposition 5(iii),

$$\hat{j}_{n+1} = \hat{j} - \sum_{i=1}^{n} \langle \hat{j}, \mathscr{B}_i \rangle \mathscr{B}_i. \tag{42}$$

By Gram-Schmidt orthonormalization process, we have $\langle \mathscr{B}_{n+1}^b, \mathscr{B}_i \rangle = 0$ and $\langle \mathscr{B}_{n+1}, \mathscr{B}_i \rangle = 0$ for each $= 1, \cdots,$. Then it follows from (42) that

$$\langle \hat{j}_{n+1}, \mathscr{B}_{n+1}^b \rangle = \langle \hat{j}, \mathscr{B}_{n+1}^b \rangle \quad \text{and} \quad \langle \hat{j}_{n+1}, \mathscr{B}_{n+1} \rangle = \langle \hat{j}, \mathscr{B}_{n+1} \rangle.$$

Hence, the equality (41) is equivalent to

$$|\langle \hat{j}, \mathscr{B}_{n+1} \rangle| = \sup \left\{ |\langle \hat{j}, \mathscr{B}_{n+1}^b \rangle| \; \middle| \; \blacktriangleright \in \mathscr{D} \setminus \{ {}_1, {}_2, \cdots, {}_n \} \right\} = S. \quad \square \tag{43}$$

5. $A^2(\mathscr{D})$

In the last section, we will prove our second main result in Theorem 2, i.e., to show a POAFD of any function $\hat{j}$ in Bergman space $A^2(\mathscr{D})$ converges to the same function $\hat{j}$.

 . For any given $\hat{j} \in A^2(\mathscr{D})$, a sequence $(\,_n)_{n\geq 1}$ of points in $\mathscr{D}$ is called a *maximal selection sequence of $\hat{j}$*, if there exists an orthonormal sequence $\{\mathscr{B}_j\}_{j\geq 1}$ in $A^2(\mathscr{D})$ associated to a sequence $(\,_n)_{n\geq 1}$ constructed as in Theorem 1 satisfying the maximal modulus property (6) in Theorem 1(iii).

For any given $\hat{j} \in A^2(\mathscr{D})$, we fix a maximal selection sequence $(\,_n)_{n\geq 1}$ of $\hat{j}$ with an orthonormal sequence $\{\mathscr{B}_j\}_{j\geq 1}$ in $A^2(\mathscr{D})$. Then Bessel's inequality implies the following

 14. *Let $\hat{j} \in A^2(\mathscr{D})$. Then for any maximal selection sequence $(\,_n)_{n\geq 1}$ of $\hat{j}$, one has* $\sum_{n=1}^{\infty} \langle \hat{j}, \mathscr{B}_n \rangle \mathscr{B}_n \in A^2(\mathscr{D})$.

Now we prove the convergence result of POAFD in Theorem 2 for any maximal selection sequence as follows.

 2. Suppose contrary that $(\,_j)_{j\geq 1}$ is a sequence given by POAFD applied to $\hat{j}$, and $\{\mathscr{B}_j\}_{j\geq 1}$ is the corresponding orthonormal sequence in $A^2(\mathscr{D})$ such that the residual

$$\sum_{j=N}^{\infty} |\langle \hat{\jmath}, \mathscr{B}_j \rangle|^2 < \left(\frac{C_0}{2} \right)^2. \tag{46}$$

By Proposition 5(ii) and inequality (46), we have

$$|\langle \hat{\jmath}_N, \mathscr{B}_N \rangle| = |\langle \hat{\jmath}, \mathscr{B}_N \rangle| < \frac{C_0}{2}. \tag{47}$$

Secondly, for any fixed $N \geq N_0$, we select a

Acknowledgments

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