SPARSE APPROXIMATION TO THE DIRAC-δ DISTRIBUTION

WEI QU, TAO QIAN*, AND GUAN-TIE DENG

ABSTRACT. The Dirac-δ distribution may be realized through sequences of convlutions, the latter being also regarded as approximation to the identity. The present study proposes the so called pre-orthogonal adaptive Fourier decomposition (POAFD) method to realize fast approximation to the identity. The type of sparse representation method has potential applications in signal and image analysis, as well as in system identification.

C A A E

Key words pr n rn r p D on r p r pr n on Ap
pro on o n o on rn Appro on rn Appro on

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Introduction

o o on p o ppro on o n n o o on o
on n r p n o on rn rn o or n r
on o on rn n r n nor on o on) n r or
o on on Fro p on n r
n ppro n n r o n on o on o on rn
o on n op n ppro on or o p
ppro on o or oo or on
$\mathcal{H}$ H_K or on) o r p n r op r o ro
n nn r pr rn q r on on or on $\mathcal{H}$ n r
r p nn r pr $\langle\cdot,\cdot\rangle_{\mathbf{H}}$. **E**, o p r r o n r
or o or o o pon n r r or o p n r **E** n
op n r p o opo o $\mathbf{R}^n$ or $\mathbf{C}^n$. r $p \in$ **E** o

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n n $h_p \in \mathcal{H}$ r o n r op r or L $\mathcal{H} \to \mathbf{C}^{\mathbf{E}}$, n o n on ro $\mathbf{E}$ o $\mathbf{C}$.

$$)\qquad L\ f)\ p) \quad \langle f, h_p\rangle_{\mathbf{H}}.$$

o r $F\ p)$ $Lf\ p)$ no $R\ L)$ n on o $F \in \mathbf{C}^{\mathbf{E}}$ o $f \in \mathcal{H}$ r pp n L. $N\ L)$ n p o n

$$N\ L) \quad \{f \in \mathcal{H} \mid L\ f) \quad \}.$$

o o $N\ L)$ o n $\mathcal{H}$. r or o on o p n o $N\ L)$ n $\mathcal{H}$ no $N\ L)^{\square}$,

$$\mathcal{H} \quad N\ L) \oplus N\ L)^{\square}.$$

A n $f \in \mathcal{H}$ n q r n

$$f \quad f^{-} \quad f^{+},$$

r $f^{-} \in N\ L), f^{+} \in N\ L)^{\square}$. n pp n no on r $L\ N\ L)^{\square})$ $R\ L)$. D no or o on pro on op r or ro $\mathcal{H}$ o $N\ L)^{\square}$ P $P\ f)$ f^{+}. n n r p r r H_K, on n on $R\ L)$, o o nor o F $L\ f)$ n r n $R\ L)$ n

$$\|F\|_{H_K} \triangleq \|Pf\|_{\mathbf{H}}.$$

nor n on n nn r pr n $R\ L)$ no $\langle \cdot, \cdot \rangle_{H_K}$. n r p H_K, o n n $R\ L)$ n or n o r $N\ L)^{\square}$ ro pp n L. n no on n on $K\ p, q)$ n

$$K\ q, p) \quad \langle h_q, h_p\rangle_{\mathbf{H}}$$

n r pr n rn o H_K, n r r pr n rn r p For proo o or $\mathcal{H}$ H_K or on on n n o n r op r or or n r p n n r p p r on n r n p r r r o n on $\{h_p\}_{p\square\mathbf{E}}$ n p o $\mathcal{H}$. n n p $N\ L)$ r on n n on ro n on n

$$\langle f, h_p\rangle \qquad \forall p \in \mathbf{E}$$

on

$$L\ f)\ p) \qquad \forall p \in \mathbf{E},$$

$N\ L)$ $\{\ \}$ $N\ L)^{\square}$ $\mathcal{H}$ p n$\{h_p\}_{p\square\mathbf{E}}$. n o r n n r o $\mathcal{H}$ no H_K n o r or r r pp n L. n o $\{h_p\}_{p\square\mathbf{E}}$ n $\mathcal{H}$ o n $\mathcal{H}$ r p on r $\{h_p/\|h_p\|\}_{p\square\mathbf{E}}$. n n n p r p r or on r q n r n o prop r o r or n n p r p p o $\mathcal{H}$ H_K r r $\mathcal{H}$ $L^2\ \partial\mathbf{D})$, L^2 p on n r $\mathbf{E}$ $\mathbf{D}, h_p\ e^{it})$ $\frac{1}{1-\overline{p}e^{\ t}}, p \in \mathbf{D}$. n $N\ L)^{\square}$ $H^2_+\ \partial\mathbf{D})$ $R\ L), N\ L)$ $H^2_-\ \partial\mathbf{D})$, n r p o r p n o n r $\mathcal{H}$ H_K or on p or o r p on no or r r po n r op r or n r p rn p r r n p r r n n r r n n r op r or p r n p r o on o o o pr o op r or n ro nn r pr rn n $\mathcal{H}$ H_K or on

o o r o n p r r p r r pr n on n n r r p po n on r o *boundary vanishing condition* B C) p r r pr n on pr or o on p Fo r o po on or AFD r §) AFD n r on o o p Fo r o po on or AFD or n op or o p H^2 p AFD or on o p r r n o r or o B r n or n o n B pr) o n n r n pp on o AFD o or n n n o n r on o AFD o n on r B o o B pr or n q n on n r on o AFD o n o r n p or o n on or n n on p o r n p o r r or po AFD n r pp r o AFD n p n o op on r o p o r or) AFD on r on on p op on AFD n p r r p por r p n on o p r r n o n n n r or o on on o r on r o on r n

n n $\mathcal{H}$ H_K or on n on $\{h_p/\|h_p\|_{\mathbf{H}}\}_{p\in\mathbf{E}}$ on r o r n p $\mathcal{H}$, n r o q n ppro o on r AFD p p r r pr n on n $\mathcal{H}$. n r pp on o AFD n $\mathcal{H}$ n on r prop r o r o p r or p r r pr n on n H_K, n n on n n o nor rn o pro B C A r n p r r p n on n H_K on r p n on o $\mathcal{H}$ ro o r pp n L^{-1}. p rpo o op n r p r r pr n on o o or D r on r n p ro po n o o $\mathcal{H}$ H_K or on n p r r ppro r n n p n n r r n o

n § r p on o AFD n § op on o on p p r r pr n on o n n r n p $\mathbf{R}^d$ n AFD n n o on rn n r on o on rn n non n r n r n n n § op o $\mathbf{E}$ p r o on p r ppro on o n n r p on o AFD n n r r p on r n B C n § o n § § n o p p r r r on o pp o o on por n o on rn on o on rn n n r p r o on rn r on r pro r r or o $\mathcal{H}$ H_K or on § on n o r p on r p p r o on AFD on p r rn AFD n $\mathbf{R}^d$.

POAFD in Hilbert Space with a Dictionary

r on p n n AFD on pr n p o r n n on on p rn r pp n or on o n n r no o n n r pro n ro q n o r A nn n AFD n or p r r pr n on o n on r n n opo o n or r o p o n n on o r n p) no n o r on p n r p on r n $\mathcal{H}$ H_K or on H_K rn n on $K(q,p)$ $K_q(p)$ $\langle h_q, h_p \rangle_{\mathbf{H}}$. n $\langle F, K_q \rangle$ or $q \in \mathbf{E}$ p F , no n on $\{K_q\}_{q \in \mathbf{E}}$ n n H_K.

Definition 2.1. *A subset $\mathcal{E}$ of a general Hilbert space H is said to be* on r *if* $\|E\|$ *for $E \in \mathcal{E}$, and* $\overline{\text{p n}}\{E \quad E \in \mathcal{E}\}$ H.

no on o on nor r pr n rn E_q $K_q/\|K_q\|, q \in \mathbf{E}$, on on r o H_K. n $\mathcal{H}$ p n n n on $h_p/\|h_p\|, p \in \mathbf{E}$, on on r o $N(L)$ $\{h_p\}_{p \in \mathbf{E}}$ n n $\mathcal{H}$, n n on $h_p/\|h_p\|, q \in \mathbf{E}$, on on r o $\mathcal{H}$.

AFD or r p on r r r on r ro r pr n rn or no o o orro no on $K_q, q \in \mathbf{E}$, no n r r pr n prop r on n n o $\{K_q\}_{q \in \mathbf{E}}$. nor or E_q $K_q/\|K_q\|$ on n n o n o o r n n on on BVC o) B or n on pr n p o AFD n o n o on p *boundary vanishing condition* BVC) *multiple reproducing kernel* BVC n o o on n on n $\mathbf{E}$ n o n r n p $\mathbf{R}^{d+1}$. n o on rn n o o n on n $\mathbf{E}$ pp r p o $\mathbf{R}^{d+1}$. n on or po n ∞, o o p $\mathbf{R}^{d+1}$. ∞ n o r po n o $\mathbf{E}$ n opo o o $\mathbf{R}^{d+1}$ ro n n op n n or o ∞ A O n op n n or o ∞ on o p n o O, O^c o p o $\mathbf{R}^{d+1}$. o p on o $\mathbf{R}^{d+1}$ r p o po n ∞. no $\partial^*\mathbf{E}$ $\partial\mathbf{E} \cup \{\infty\}$, o o r po n o $\mathbf{E}$ n n opo o p $\mathbf{R}^{d+1} \cup \{\infty\}$, r $\partial\mathbf{E}$ o n o r po n o $\mathbf{E}$. A on q n n op n n o r o $\partial^*\mathbf{E}$ n on o n op n n o r o $\partial\mathbf{E}$ n op n n or o ∞. n r on po n o p on opo o p $\mathbf{R}^{d+1} \cup \{\infty\}$ o p o $\mathbf{E} \cup \partial^*\mathbf{E}$ o o p

$\mathbf{E}$ o op n n $\mathbf{R}^{d+1}$, n p r o on rn ppro on n o no o n n or n opo o n r on n on $\partial^*\mathbf{E}$ $\partial\mathbf{E}$. Bo r n n on BVC) o o o $\mathbf{E}$ r

Definition 2.2. *Let H be a Hilbert space with a dictionary $\mathcal{E}$ $\{E_q\}_{q \in \mathbf{E}}$. If for any $f \in \mathcal{H}$ and any $q_k \to \partial^*\mathbf{E}$, in the one-point-compactification topology if necessary, there holds*

$$\lim_{k \to \infty} |\langle f, E_{q_k} \rangle| \quad ,$$

then we say that H together with $\mathcal{E}$ satisfy BVC.

H r p on r $E_q, q \in \mathbf{E}$, n B C n o p r n o on on or n $f \in H$ r on o $\tilde{q} \in \mathbf{E}$

$$\tilde{q} \quad r \quad p\{|\langle f, E_q\rangle|^2 \quad q \in \mathbf{E}\}.$$

no r p H n n on r o $\mathcal{H}$ H_K o H $\mathcal{H}$ or H H_K. H $\mathcal{H}$ r r o o n r $\{h_p/\|h_p\|\}_{p \in \mathbf{E}}$, n on H H_K r r o o on o nor r pr n rn E_q $K_q/\|K_q\|, q \in \mathbf{E}$.

n n n p B r n B r n p B C n o r r o nor r pr n rn on n on r o no B C D)

n p rn $q_1, \cdots, q_n$ n n p o p r r n $\mathbf{E}$. $\mathbf{E}$ r on n o p n or on n $\mathbf{R}^d$, or n n $\mathbf{C}^d$. n $\mathbf{R}^d$

$$\tilde{K}_{q_n}(p) \quad \left[\left(\frac{\partial}{\partial q_{\vec{\theta}}}\right)^{j(n)-1} K_q\right]_{q=q_n}(p), \qquad)$$

r $j(n)$ n r o r p o p r r n n p $q_1, \cdots, q_n$), n $\frac{\partial}{\partial q}$ $\vec{\theta} \cdot \nabla$, n r on r n on $\vec{\theta}$ o $\theta_1, \cdots,$ o θ_d).

$\mathbf{E} \subset \mathbf{C}^d$, on p r n Fo $\mathbf{E}$ $\mathbf{C}$ r on r p r p $e^{i\theta}\frac{\partial}{\partial z}$. no p n q_k / q_n or $k < n$, n $j(n)$, n $\tilde{K}_{q_n}$ K_{q_n}. n $\tilde{K}_{q_n}(p)$ *multiple kernel* o n p $q_1, \cdots, q_n$). A o n n q n $q_1, \cdots, q_n, \cdots$),

r n n n n q n o p n

n AFD or or g g_n n n r r

$$g_n \quad f - \sum_{k=1}^{n-1}\langle f, B_k\rangle B_k.$$

n on r o n r por on ro
r r A p p op on or AFD r
op on r r n B C on p p rn
o on o po on o AFD o n r r

Remark 2.3. *H* o no on r n B C n n p rn on nno p r or AFD o r ro n on o pr or n $\rho \in$,) n n $q_1, \cdots, q_{n-1}$, r q r n ro pr n q_k, k $, \cdots, n-$,

$$) \qquad |\langle g, B_n^{q_n}\rangle| \geq \rho \quad \text{p}\{\langle g, B_n^{q}\rangle| \quad q \in \mathbf{E}, q \;/\; q_1, \cdots, q_{n-1}\}.$$

or or on n q_n, n $, \cdots, n, \cdots,$ o) *Weak Pre-orthogonal adaptive Fourier decomposition* AFD) r o n op AFD pr n p n nn r n r p p r r n ro n o n n pr o p or o r r or n r n r n o r q_n o) r n n p AFD q n AFD n pro r no on or onor o r or onor on o p o rn o on

B n AFD on n pro n r r o AFD o n n $M/\sqrt{n}$ p n on on o p

$$H^M \quad \{f \mid f \in H, \exists\, q_k, d_k \quad f \quad \sum_{k=1}^{\infty} d_k E_q \quad \sum_{k=1}^{\infty} |d_k| \leq M\}$$

)

r r o on r n r on pro n r no oo n on on po o p n on on r p or r on r n r r A n n AFD o pro $\mathcal{H}$ H_K or on n n r o on o n r r n q on) n pr n p p r on p or p p r r pr n on o D r on)

Sparse Approximation of the Convolution Type

Sparse Poisson Kernel Approximation. no n o on n r ppro o r n on n on op p r ppro on n r o n on o p r r o on rn o on rn on $\mathcal{H}$ H_K or on $\mathcal{H}$ L^2 $\mathbf{R}^d$).

$$\mathbf{E} \quad \{p \in \mathbf{R}_+^{d+1} \mid p \quad t \quad \underline{x}, t > \;, \underline{x} \quad x_1, \cdots x_d)\}.$$

$+$

For p t $+$ $\underline{x}$,

$$h_p\ \underline{y})\quad P_{t+\underline{x}}\ \underline{y}) \triangleq c_d \frac{t}{|p-\underline{y}|^{d+1}}\quad c_d \frac{t}{t^2 + |\underline{x}-\underline{y}|^2)^{\frac{+1}{2}}},\quad d \geq\ ,$$

r c_d $\frac{\Gamma[(d+1)/2]}{\pi^{(\ +1)\ 2}}$. no $h_p\ \underline{y})$ on po n $\underline{x}-\underline{y}$ o L^1 on o n on $\phi\ \underline{x})$ $c_d \frac{1}{(1+|\underline{x}|^2)^{(\ +1)\ 2}}$, r c_d nor on on n r n r o ϕ o r $\mathbf{R}^d$ n . n o n o on rn on n p r rn n $\mathbf{R}^d$ c_d or nor on p rpo o n r ro on o on op r or L $Lf, f \in L^2\ \mathbf{R}^d)$, r

$$u\ t + \underline{x})\quad Lf\ t + \underline{x})\quad \langle f, h_{t+\underline{x}}\rangle_{L^2(\mathbf{R}\ )}.$$

n $\mathcal{H}$ H_K or on r n $R\ L)$ on o o on n r o o r $f \in L^2\ \mathbf{R}^d)$. o o $\{h_p\}_{p\in\mathbf{E}}$ n o o $f \in L^2\ \mathbf{R}^d)$ $\langle f, h_p\rangle$ or p, n f . r o r on n n o $L^2\ \mathbf{R}^d)$ nor po n n

$$\lim_{t\to 0+} u\ t + \underline{x})\quad \lim_{t\to 0+}\langle f, h_p\rangle\quad f\ \underline{x})\quad ,\quad . .$$

n $\mathcal{H}$ H_K or on $N\ L)$ $\{\ \}$ $N\ L)^{\perp}$ $L^2\ \mathbf{R}^d)$. n o r $R\ L)$ H_K r pp n L o r $N\ L)^{\perp}$ $L^2\ \mathbf{R}^d)$. n p r r $L\ h_q)$ K_q. D n o K_q n H_K p n o h_p n $\mathcal{H}$ $L^2\ \mathbf{R}^d)$. r on n no n r r on o p H_K. n H_K o n o r nor r on p on pp r p $\mathbf{R}_+^{d+1}$

$$h^2\ \mathbf{R}_+^{d+1})\quad \{u\quad \mathbf{R}_+^{d+1} \to \mathbf{R}\quad \triangle_{\mathbf{R}_+^{\ +1}} u\quad , \|u\|^2_{h^2(\mathbf{R}_+^{\ +1})}\quad \sup_{t>0} \int_{\mathbf{R}} |u\ t + \underline{x})|^2 d\underline{x} < \infty\}.$$

B no n $f\ \underline{x})$ u $+$ $\underline{x})$,

$$\|u\|^2_{H_K}\ \mathsf{H}\ {}^{H_K}\ \|f\|^2_{L^2(\mathbf{R}\ )}\ {}^{Nd}$$

r t $\underline{x}$ o r n on $P_{t_1+\underline{x}_1}(\cdot)$. r or o r on n on

o on o on r on

$$\langle P_{t_1+\underline{x}_1}, P_{t+\underline{x}}\rangle_{L^2(\mathbf{R})} \quad P_{(t_1+t)+(\underline{x}_1-\underline{x})} \quad),$$

r ro p prop r o o on rn r p n prop r o K_q n on q n o $\mathcal{H}$ H_K or on For $u \in H_K, q$ t_1 $\underline{x}_1$,

$$\langle u, K_q\rangle_{H_K} \quad \langle u \quad (\cdot), P_{t_1+\underline{x}_1}(\cdot)\rangle_{L^2(\mathbf{R})} \quad u \; t_1 \quad \underline{x}_1).$$

For n r K_q, q t $\underline{x}, t >$, nor o p ro ro p prop r)

$$\|K_q\|^2_{H_K} \quad \langle K_q, K_q\rangle_{H_K} \quad K\; q, q) \quad P_{2t} \quad) \quad \frac{c_d}{t)^d}.$$

nor on nor on o K_q

$$E_q \quad \frac{K_q}{\|K_q\|} \quad \left(\frac{t)^d}{c_d}\right)^{1/2} K_q.$$

r B C n o on on

$$) \qquad \lim_{q\to\partial\mathbf{E}} |\langle u, E_q\rangle_{H_K}| \qquad ,$$

r u n n on n H_K h^2 $\mathbf{R}^{d+1}_+$). r n r p n rn prop r or q t $\underline{x}$,

$$) \qquad \langle u, E_q\rangle_{H_K} \quad c_d' t^{d/2} u \; t \quad \underline{x}).$$

D n o p r r o on rn n H_K, r on o B C o r n) or p r r r p n rn $u\; p)$ $K_{t_1+\underline{x}_1}$. Fro)

$$) \; \langle K_{t_1+\underline{x}_1}, E_q\rangle_{H_K} \quad c_d t^{d/2} P_{(t_1+t)+\underline{x}_1} \; \underline{x}) \quad c_d' t^{d/2} \frac{t \quad t_1}{t \quad t_1)^2 \quad |\underline{x} - \underline{x}_1|^2 \; (d+1)/2}.$$

n pro $q \to \partial' \mathbf{R}^{d+1}_+$ on on po n o p on opo o o n o rn $t \to$ or t^2 $\underline{x}^2 \to \infty$. For n $\underline{x}_1$ $t_1 >$, r po on o $\underline{x}$,

$$t^{d/2} \frac{t \quad t_1}{t \quad t_1)^2 \quad |\underline{x} - \underline{x}_1|^2 \; (d+1)/2} \le t^{d/2} \frac{}{t \quad t_1)^d} \to \quad ,$$

$t \to$ $d \ge$) o n or n $\underline{x}$, $t \to$, q n n) o ro or t_1 $\underline{x}_1$, R $\sqrt{t^2 \quad |\underline{x}|^2} >$ $|\underline{x}_1|$ t_1 . r n o o) $< t < R/$) $t \ge R/$. n) $|\underline{x}| > R/$ n $\underline{x} - \underline{x}_1|)^2 \ge$ $|\underline{x}|/$ $)^2$. n nor n on n

$$t)^{d/2} \frac{t \quad t_1}{t \quad t_1)^2 \quad |\underline{x} - \underline{x}_1|^2 \; (d+1)/2} \le \; R/ \;)^{d/2} \frac{R/ \quad R/}{t \quad t_1)^2 \quad |\underline{x}/ \;|)^2 \; (d+1)/2} \le \frac{c_d}{R^{d/2}} \to \quad ,$$

$R \to \infty$ $d \ge$)

) $t \ge R/$ p

$$t^{d/2} \frac{t \quad t_1}{t \quad t_1)^2 \quad |\underline{x} - \underline{x}_1|^2 \; (d+1)/2} \le t^{d/2} \frac{}{t \quad t_1)^n} \le \frac{c_d'''}{R^{d/2}} \to \quad ,$$

$R \to \infty$ $d \geq$) n or n t, n $\underline{x}$ o n n q n n)
o ro
B C pro r o) n n **E**. n AFD n
pro

Sparse Heat (Gaussian) Kernel Approximation. no n n r
r n or on

$$) \quad Lf\, t\ {}_{+}\ \underline{x}) \quad \frac{}{\pi t)^{d/2}} \int_{\mathbf{R}} f\ \underline{y}) e^{-\frac{|\underline{x}-\underline{\ }|}{4t}} d\underline{y}, \quad n \geq\ ,$$

r o n q o on $u\ t\ {}_{+}\ \underline{x})$ o n pr or q on

$$\frac{\partial u}{\partial t} \quad \triangle u, \quad u\ {}_{+}\ \underline{x}) \quad f\ \underline{x}), \quad f \in L^2\ \mathbf{R}^d),$$

r $\triangle$ p n or $x_1, \cdots, x_d$. n o $\mathcal{H}$ H_K or on
$\mathcal{H}$ L^2 $\mathbf{R}^d), q$ $t\ {}_{+}\ \underline{x} \in \mathbf{E}$ $\mathbf{R}_+^{d+1}$,

$$) \quad h_q\ \underline{y}) \quad \frac{}{\pi t)^{d/2}} e^{-\frac{|\underline{x}-\underline{\ }|}{4t}}.$$

rn n r $\{h_p\}_{p \in \mathbf{R}_+^{+1}}$ n n L^2 $\mathbf{R}^d)$. p
N $L)$ n r p on n o on ro n on p
N $L)$ o n L^2 $\mathbf{R}^d)$. n $\mathcal{H}$ H_K or on p H_K r n
R $L)$ q pp nor ro r L^2 o r A o
rn p H_K r r n n on or h^2 $\mathbf{R}_+^{d+1})$
n o on rn n q n on on p on
o on o n n r n q on r o no p or or
r no or n p rpo o p L^2 $\mathbf{R}^d)$ o n
non n n o r B) o r r n r u Lf

$$\lim_{t \to 0} u\ t\ {}_{+}\ \underline{x}) \quad f\ \underline{x}), \quad f \in L^2\ \mathbf{R}^d),$$

n o L^2 $\mathbf{R}^d)$ po n n For r on no $f\ \underline{x})$
$u\ {}_{+}\ \underline{x})$, n o on n r r on

$$\|u\|^2_{H_K} \quad H_K \quad \|f\|^2_{L^2(\mathbf{R}\ )} \quad NBL \quad \|u\ {}_{+}\ \cdot)\|^2_{L^2(\mathbf{R}\ )}.$$

For rn p H r r r on or o on
rn n r on h^2 p For o r ppro on p rpo on
r pr n rn q $t\ {}_{+}\ \underline{x}$ p $s\ {}_{+}\ \underline{y}$

$$K\ q, p) \quad \langle h_q, h_p \rangle_{L^2(\mathbf{R}\ )}$$

$$) \quad \frac{}{\pi)^d\ ts)^{d/2}} \int_{\mathbf{R}} e^{\frac{-|\underline{x}-\underline{\ }|^2}{4t}} e^{\frac{-|\underline{\ }-\underline{\ }|^2}{4s}} d\xi.$$

n r r pr n n $K\ q, p)$ q o

$$) \quad \frac{}{\pi)^d\ ts)^{d/2}} \int_{\mathbf{R}} e^{\frac{-|\underline{x}-\underline{\ }|^2}{4t}} e^{\frac{-|\underline{\ }-\underline{\ }|^2}{4s}} d\xi \quad \frac{}{\pi\ t\ {}_{+}\ s))^{d/2}} e^{\frac{-|\underline{x}-\underline{\ }|^2}{4(t+s)}},$$

r or n o)

$$) \quad K\ q, p) \quad h_{(t+s)+\underline{x}}\ \underline{y}) \quad h_{(t+s)+\underline{y}}\ \underline{x}) \quad h_{(t+s)+(\underline{x}-\underline{y})}\ ).$$

proo o n) n o on rn
n o n pro

$$\frac{\partial u}{\partial t} \quad \triangle u, \quad \lim_{s\to 0+} u\ s \quad {}_{+}\ \underline{y}) \quad h_{(t+0)+\underline{x}}\ \underline{y}).$$

r on

$$\langle h_{t+\underline{x}}, h_{s+\underline{y}}\rangle_{L^2(\mathbf{R}\ )} \quad P_{(t+s)+(\underline{x}-\underline{y})} \quad),$$

r ro p prop r o rn
o pro rn p r r pr n on n AFD r r B C
nor o rn K_q o p ro

$$\|K_q\|^2_{H_K} \quad \langle K_q, K_q\rangle_{H_K} \quad h_{2t+\underline{0}}\ \_) \quad \frac{}{\pi t)^{d/2}}.$$

n nor r pr n rn K_q or q t $_+$ $\underline{x}, p$ s $_+$ $\underline{y}$ o

$$) \qquad E_q\ p) \quad \pi t)^{d/4} h_{(t+s)+\underline{x}}\ \underline{y}) \quad \frac{\pi t)^{d/4}}{\pi\ t\ {}_{+}\ s))^{d/2}} e^{-\frac{|\underline{x}-\_|^2}{4(t+s)}}.$$

r o o B C

$$\sup_{\mathbf{R}_+^{+1} \ni q \to \partial \mathbf{R}_+^{+1}} |\langle$$

n) $t \geq R/$. ro r on on)

$$\pi t)^{d/4} h_{(t+s)+\underline{x}}\ \underline{y}) \leq C \left(\frac{}{t \quad s} \right)^{d/4} \leq C \frac{}{R^{d/4}} \to \ , \qquad R \to \infty.$$

+

n or n r n r on or p r r r pro n rn n on n n r n r n) o o on rn o n on n or o on rn r or p s $\underline{y}$ q t $\underline{x} \to \partial \mathbf{R}^{d+1}_{+}$, o $+$ $+$

$$|\langle K_p, E_q\rangle_{H_K}| \leq Ct^{d/2} P_{(s+t)+\underline{y}}\ \underline{x}) \to \ .$$

For $< \delta <$ o no no n r n no AFD pp o r o o on $\mathcal{H}$ on or H_K on n H_K on r pr n rn prop r o r on n n p n rn n p or

Poisson Kernel Sparse Approximation on Spheres

$\mathcal{H}$ L^2 $\mathbf{S}^{d-1})$, r p o q r n n on on $d-$ n on n p r n or n $\mathbf{E}$ $\mathbf{B}^d$, n on op n n n or n r $d \geq$. For po n $q \in \mathbf{B}^d$, n on h_q n on o on rn o q rt, r $|q|, t, s \in \mathbf{S}^{d-1}$,

$$) \qquad h_q\ s) \quad P_q\ s) \quad c_d \frac{-r^2}{|q-s|^d}.$$

op r or L $Lf, f \in L^2$ $\mathbf{S}^{d-1})$, r

$$u\ q) \quad Lf\ q) \quad \langle f, h_q\rangle_{L^2(\mathbf{S}\ \ -1)},$$

r nn r pr o L^2 $\mathbf{S}^{d-1})$

$$\langle f, g\rangle_{L^2(\mathbf{S}\ \ -1)} \quad \int_{\mathbf{S}\ \ -1} f\ s) g\ s) d\sigma\ s),$$

r $d\sigma$ $s)$ nor r on p r r n R $L)$ r on p on $\mathbf{B}^d$

$$h^2\ \mathbf{B}^d) \quad \{u \quad \mathbf{B}^d \to \mathbf{R} \quad \triangle u \quad , \ \underset{0 \leq r<1}{\mathrm{p}} \int_{\mathbf{S}\ \ -1} |u\ rs)|^2 d\sigma\ s) < \infty\}.$$

D o n o $\{h_q\}_{q \in \mathbf{B}}$ n L^2 $\mathbf{S}^{d-1})$ p N $L)$ r on n o on ro n on p N $L)^{\perp}$ n L^2 $\mathbf{S}^{d-1})$. p H_K h^2 $\mathbf{B}^d)$, n r n R $L)$ q p nor ro r non n n o r For u Lf,

$$\|u\|_{H_K} \triangleq \|f\|_{L^2(\mathbf{S}\ \ -1)}.$$

no n no

$$\lim_{r \to 1} u\ rt) \quad f\ t)$$

o L^2 n po n n For r on r f $t)$ u $t)$ r orr po n u Lf. o r on

$$\|u\|^2_{H_K} \overset{\mathrm{H}\ H_K}{} \|f\|^2_{L^2(\mathbf{S}\ \ -1)} \overset{NBL}{} \|u\ \cdot)\|^2_{L^2(\mathbf{S}\ \ -1)} \overset{h^2\ \text{Theory}}{} \underset{0 \leq r<1}{\mathrm{p}} \int_{\mathbf{S}\ \ -1} |u\ rt)|^2 d\sigma\ t).$$

r pr n rn o p H_K o p or q rt, p $\rho s, t, s \in \mathbf{S}^{d-1}$,

$$K\ q,p) \quad \langle h_q, h_p\rangle_{L^2(\mathbf{S}\ -1)}$$

$$\int_{\mathbf{S}\ -1} h_q\ t)h_p\ t)d\sigma\ t)$$

$$\int_{\mathbf{S}\ -1} \frac{-r^2}{|q-t|^d}\frac{-\rho^2}{|p-t|^d}d\sigma\ t)$$

$$)\qquad P_{\rho rt}\ s)$$

$$P_{r\rho s}\ t).$$

q o r on $|r\rho t-s|^2$ $|r\rho s-t|^2$. o pro q o o q n n on n r r on n p, n $\rho \to$, n r o $P_q\ s)$ $P_{rs}\ t)$. n o r $P_{\rho rt}\ s)$ $P_{r\rho s}\ t)$ o r on n on n p, r ro po r oo n or o p n

$$\triangle_p \quad \frac{\partial^2}{\partial\rho^2} + \frac{d-}{\rho}\frac{\partial}{\partial\rho} + \frac{}{\rho^2}\triangle_{\mathbf{S}\ -1},$$

r $\triangle_{\mathbf{S}\ -1}$ B r p r p n on p r n

$$\triangle_p\ P_{r\rho s}\ t)) \quad r^2\left(\frac{\partial^2}{\partial\ r\rho)^2} + \frac{d-}{r\rho)}\frac{\partial}{\partial\ r\rho)} + \frac{}{r\rho)^2}\triangle_{\mathbf{S}\ -1}\right)P_{(r\rho)s}\ t) \quad .$$

r on n on or $\rho \to$ o r n on $P_{rs}\ t$ D o n q n o o on o r on o r pro on q

o r on o r p prop r o o on rn on p r pro o p on on n n $\mathcal{H}$ H_K or on r pr n prop r o K_q o For $u \in H_K$ $h^2\ \mathbf{S}^{d-1}), q$ rt,

$$\langle u, K_q\rangle_{H_K} \quad \langle u\ \cdot), P_{rt}\ \cdot)\rangle_{L^2(\mathbf{S}\ -1)} \quad u\ rt) \quad u\ q).$$

o p r or AFD n H_K $h^2\ \mathbf{B}^d)$ n o pro orr po n B C For n r K_w, w $\rho s, s \in \mathbf{S}^{d-1}$, $< \rho <$, nor o p

$$\|K_w\|^2_{H_K} \quad \langle K_w, K_w\rangle_{H_K} \quad K\ w,w) \quad P_{\rho^2 s}\ s) \quad \frac{c_d \quad \rho^2)}{-\rho^2)^{d-1}}.$$

nor on o K_q no

$$E_w \quad \frac{K_w}{\|K_w\|} \quad \frac{-\rho^2)^{(d-1)/2}}{\sqrt{c_d \quad \rho^2)}}K_w.$$

r o r B C or n n n on n H_K,

$$) \qquad {}_{\mathbf{B}\ w\to\mathbf{S}\ -1} |\langle u, E_w\rangle_{H_K}| \quad .$$

r r or w $\rho s, s \in \mathbf{S}^{d-1}$,

$$) \qquad \langle u, E_w\rangle_{H_K} \quad \frac{-\rho^2)^{(d-1)/2}}{\sqrt{c_d \quad \rho^2)}}u\ w).$$

D n o p r r p r o on rn r on o B C or n r n on $u \in H_K$ o r n B C or n p r r p r o on rn K_q, q $rt, t \in \mathbf{S}^{d-1}$. n o r pr n rn pr on)

$$) \qquad E_w\ q) \quad \langle K_q, E_w\rangle_{H_K} \quad \frac{-\rho^2)^{(d-1)/2}}{\sqrt{c_d \quad + \quad \rho^2)}} P_{r\rho t}\ s).$$

n $\rho \to -$, q n $P_{r\rho t}\ s)$ r o n or n t, s, o p on $r <$. n or n ron o ro or $d \geq$, o q n o ro n or n t, s. B C pro AFD p r or

Remark 4.1. AFD ppro on n o o n or n po n r n or n on $u \in H_K^M$ r n N o n on o p r r o on rn

$$\|u - \sum_{k=1}^{N} c_k P_q \ \|_{h^2(\mathbf{B}\ )} \leq \frac{M}{\sqrt{N}}.$$

r n r o o r n L^2 $\mathbf{S}^{d-1})$

$$\|f\ \cdot) - \sum_{k=1}^{N} c_k P_q\ \ \cdot)\|_{L^2(\mathbf{S}\ -1)} \leq \frac{M}{\sqrt{N}}.$$

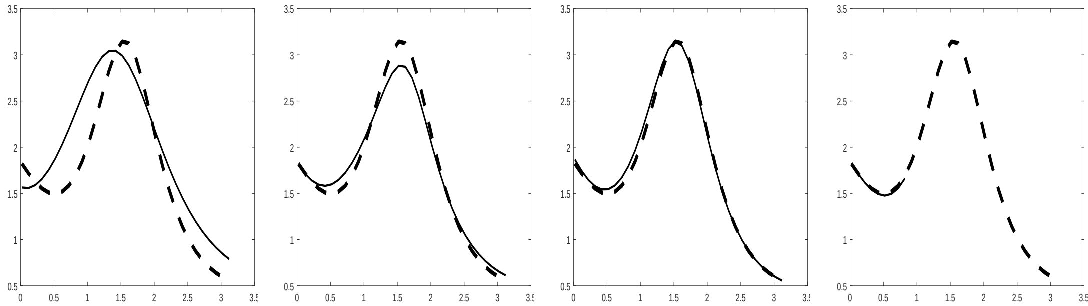
Experiments

o p r n on p r p r o on rn rn ppro on r n B o o n r pr n or n n on n r pr n ppro on n on

Example 5.1. p r p r o on Appro on) $f\ q)$ o p n r o n on o r nor p r o on rn on p r or)) E_{p_1}, E_{p_2} E_{p_3}, r $f\ q)$ $\sum_{j=1}^{3} c_j E_p\ \ q)$, r $c_1, c_2, c_3)$. , . , .), p_j $\rho_j\ \underline{s_j})$, $\rho_1, \rho_2, \rho_3)$. , . , .), $\underline{s_j}$ nϕ_j o θ_j, nϕ_j nθ_j, o $\phi_j), \phi_j \in$ $,\pi$ $, \theta_j \in$ $, \pi)$, $\phi_1, \phi_2, \phi_3)$ $\pi/$ $,\pi/$, $\pi/$), $\theta_1, \theta_2, \theta_3)$ $\pi/$, $\pi/$, $\pi/$). r

$$) \qquad f\ q) \quad \sum_{j=1}^{3} c_j \frac{-\rho_j^2}{\sqrt{\quad + \quad \rho_j^2}} \frac{-\ r\rho_j)^2}{|r\rho_j \underline{s}_j - \underline{t}|^3},$$

r q r $\underline{t}), \underline{t}$ nϕ o θ, nϕ nθ, o $\phi), \phi \in$ $,\pi$ $, \theta \in$ $, \pi)$. B n on on o r r ro) o n AFD r on n o r AFD p n on r rror r p $\times$ $^{-5}$. on p r r q_j $h_j\ \underline{w_j})$ $\underline{w_j}$ nα_j o β_j, nα nβ_j, o $\alpha_j), \alpha_j \in$ $,\pi$ $, \beta_j \in$ $, \pi), j$ $, \cdots,$, r $h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8)$. , . , . , . , . , . , . , .), $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$. , . , . , . , . , . , . , .), $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8)$. , . , . , . , . , . , . , .). For o n ppro on n o r o r n n on r $, \theta$. , r p r n on o ϕ r n n π.



2 POAFD iterations

4 POAFD iterations

6 POAFD iterations

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