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Spectra of rational orthonormal systems

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Abstract We

More speci cally, when all a_k's are zero the system reduces to the classical nite impulse response (FIR) models, a half of the Fourier system [5], namely, $\{z^k\}_{k=0}^{\infty}$, a basis of the Hardy spaces $H^p(\mathbb{D}), 1 < p < \infty$. For general parameters a_k is a system $\{E_{\vec{a}_n}\}_{n=0}^{\infty}$ which is the basis of a Hardy H^p space for $p \in (1, \infty)$ only if the hyperbolic non-separability condition is satis ed. For $p = 2$, a TM system is always an orthonormal system of $H^2(\mathbb{D})$, regardless of whether it is a basis, i.e., regardless of whether the hyperbolic non-separability condition is met.

Such systems have been well studied in pure mathematics and applied science [2,3,6,8,16,19,20,23,24]. In recent decades, engineers have paid a signi cant amount of attention to rational orthogonal systems in the unit disc case [1, 12{14, 18]. For system identi cation of stable linear time-invariant systems, it is crucial, in practice, to form dynamical models from measured data. Rational model structures, such as the ARX (autoregressive with exogenous terms) and ARMAX (autoregressive moving-average with exogenous terms) models are natural choices, because almost all systems can be described by rational transfer functions (Hardy space functions) [9, 10]. Recently, researchers in time-frequency analysis have shown an increasing interest in rational orthogonal systems to investigate analytic signals from a nonlinear phase in $L^2(\partial\mathbb{D})$ or $H^2_{\pm}(\partial\mathbb{D})$ with a positive instantaneous frequency. Here, $H^2_+(\partial\mathbb{D})$ and $H^2_-(\partial\mathbb{D})$ consist of the non-tangential boundary limits of the complex Hardy H^2 functions inside and outside the unit disc, respectively. Due to $E_{\vec{a}_n} \in H^2(\mathbb{D})$ and taking the non-tangential boundary limits of $E_{\vec{a}_n}$, we get the boundary TM systems $\{e_{\vec{a}_n}\}_{n=0}^{\infty} \subset H^2_+(\partial\mathbb{D})$ de ned by

$$e_{\vec{a}_n}(t) := \lim_{r\to 1^-} E_{\vec{a}_n}(r\mathrm{e}^{\mathrm{i}t}), \quad t \in \mathbb{T} = [-\pi, \pi] \tag{1.3}$$

for some vector $\vec{a}_n = (a_0, \dots, a_{n-1}, a_n)^{\mathrm{T}} \in \mathbb{D}^{n+1}$, and in particular, the Laguerre and the Kautz systems

$$e_n(t) := \lim_{r\to 1^-} E_n(r\mathrm{e}^{\mathrm{i}t}), \quad t \in \mathbb{T}. \tag{1.4}$$

The system $\{e_{\vec{a}_n}\}_{n=0}^{\infty}$ consisting of basic functions of nonlinear phases is an orthonormal basis of $H^2_+(\partial\mathbb{D})$ if the hyperbolic non-separability condition is met. Here, a nonlinear phase is such that each $e_{\vec{a}_n}$ has the polarized factorization $e_{\vec{a}_n}(t) = \rho_n(t)\mathrm{e}^{\mathrm{i}\theta_n(t)}$ with the nonlinear phase function θ_n. Since the space $L^2(\partial\mathbb{D})$ can be expressed as the direct sum of the two relevant boundary Hardy spaces, namely, $L^2(\partial\mathbb{D}) = H^2_+(\partial\mathbb{D}) \oplus H^2_-(\partial\mathbb{D})$, the system $\{e_{\vec{a}_n}\}_{n=0}^{\infty} \cup \{\overline{e_{\vec{a}_n}}\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(\partial\mathbb{D})$ if the two systems are bases in their respective spaces.

The fundamentality of the Fourier system demonstrated by its explicit representations, generality, and e ectiveness depends, to a large extent on the general entries of the system. Research results on TM systems and their general terms, namely, nonlinear Fourier atoms can be regarded as advances of Fourier theory. They include a Bedrosian identity, nonlinear phase basis, adaptive algorithm (see [4, 15, 17, 18]).

Notice that the operator $\frac{d^2}{dt^2} : C^2[-\pi, \pi] \to C[-\pi, \pi]$ has a discrete spectrum, a discrete set of the eigenfunctions $\mathrm{e}^{\mathrm{i}n\cdot}$, that is the Fourier system. (Notice that if the domain is changed to $\mathbb{R}$, the spectrum will be continuous and $\mathrm{e}^{\mathrm{i}x\cdot}$ will be the generalized eigenvectors.) It is natural to ask whether any general rational system would have a spectral operator $\mathcal{L}$. This note gives a part of the answer to this question. We will use the Weyl correspondence theory to investigate the spectral operators of the Laguerre systems and the Kautz systems $\{e_n\}_{n=0}^{\infty}$. Concretely, we will look for $F \in \mathcal{S}'(\mathbb{R}^2)$ to generate a di erential operator $\mathcal{L}$ through the Weyl transform (2.2) such that e_n is the eigenvector of $\mathcal{L}$. We also generalize the results to the cases of multiple parameters with a complex variable in both the unit disc and the upper half-plane contexts.

We found that the spectral operators of the Laguerre systems and the Kautz systems are closely related to the general Sturm-Liouville operators. We will deal with the general Sturm-Liouville operators under the framework of Heisenberg group and Weyl correspondence.

Section 2 reviews the Heisenberg group and Weyl correspondence, and then discusses how to generate the Sturm-Liouville operators. Section 3 focuses on the spectral operator of the Laguerre systems and the Kautz systems. Section 4 generalizes the results of Section 3 to the multiple-parameter cases through the one complex variable setting. Section 5 considers the upper half-plane. Section 6 discusses the Cayley transformation method that converts the results of the upper half-plane to the unit disc and vice versa.

2 Weyl correspondence and Sturm-Liouville operators

The group representation of Heisenberg group suggests an exponential type operator $e^{2\pi i(p\mathcal{D}+q\mathcal{X})}$ (see (2.1)) and then leads to the Weyl correspondence theory to generate pseudo-di erential operators. The full Heisenberg group is the set $(\mathbb{R}^n)^2 \times \mathbb{R}$ with the multiplication

$$(p, q, d)(p', q', d') = \left(p + p', q + q', d + d' + \frac{1}{2}[(p, q), (p', q')]\right),$$

where the symplectic product on the phase space $(\mathbb{R}^n)^2$ is de ned by $[(p, q), (p', q')] = pq' - qp'$. For $p, q \in \mathbb{R}^n, d \in \mathbb{R}$, denoted by $\mathcal{R}_d$, $\mathcal{M}_q$ and $\mathcal{T}_p$, the usual rotation, modulation, and translation operators are de ned, respectively, by

$$\mathcal{R}_d f(x) = e^{\pi i d} f(x), \quad \mathcal{M}_q f(x) = e^{2\pi i q x} f(x), \quad \mathcal{T}_p f(x) = f(x - p), \quad x \in \mathbb{R}^n.$$

The Schrodinger representation of the full Heisenberg group is the unitary operator $\mathcal{R}_d \mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}$. Up to the rotation factor $\mathcal{R}_d$, the symmetric form $\mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}$ is crucial in harmonic analysis in the phase space. We adopt Folland's notation [7]:

$$\rho(p, q) = e^{2\pi i(p\mathcal{D}+q\mathcal{X})} = \mathcal{M}_{\frac{q}{2}} \mathcal{T}_{-p} \mathcal{M}_{\frac{q}{2}}. \tag{2.1}$$

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q998J.004(q998JT106.974Tf6.1223.63.9Td(2)TiDq1001376.275328446)Ti30m3.4614239SBT/T11753284454832

Weyl's prescription for assigning an operator $\sigma(\mathcal{D},\mathcal{X})$ to a function $\sigma(\xi,x)$ amounts to postulating that the exponential function $\mathrm{e}^{2\pi\mathrm{i}(p\xi+qx)}$ should correspond to the operator $\rho(p,q)=\mathrm{e}^{2\pi\mathrm{i}(p\mathcal{D}+q\mathcal{X})}$ defined in (2.1). Once this is granted, one can expand an arbitrary $\sigma(\xi,x)$ in terms of an exponential via the inverse Fourier transform $\sigma(\xi,x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{\sigma}(p,q)\mathrm{e}^{2\pi\mathrm{i}(p\xi+qx)}dpdq$ to create a new operator

$$\sigma(\mathcal{D},\mathcal{X})=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{\sigma}(p,q)\mathrm{e}^{2\pi\mathrm{i}(p\mathcal{D}+q\mathcal{X})}dpdq=\rho(\hat{\sigma}). \tag{2.4}$$

This integral is a Bochner integral if $\hat{\sigma}\in L^1(\mathbb{R}^{2n})$. Proposition 2.1 indicates that the notation $\sigma(\mathcal{D},\mathcal{X})=\rho(\hat{\sigma})$ makes sense as an operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ whenever $\hat{\sigma}$ and σ are both tempered distributions.

From (2.3), $\sigma(\mathcal{D},\mathcal{X})=\rho(\hat{\sigma})$ is an integral operator whose distribution kernel is

$$\begin{aligned}K_\sigma(x,y)&=(\mathcal{F}_2^{-1}\hat{\sigma})\left(y-x,\frac{y+x}{2}\right)=(\mathcal{F}_1\sigma)\left(y-x,\frac{y+x}{2}\right)\\&=\int_{\mathbb{R}^n}\sigma\left(\xi,\frac{x+y}{2}\right)\mathrm{e}^{2\pi\mathrm{i}(x-y)\xi}d\xi.\end{aligned}$$

As a consequence, the operator $\sigma(\mathcal{D},\mathcal{X})=\rho(\hat{\sigma})$ is given by

$$\sigma(\mathcal{D},\mathcal{X})f(t)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma\left(\xi,\frac{t+y}{2}\right)\mathrm{e}^{2\pi\mathrm{i}(t-y)\xi}f(y)dyd\xi. \tag{2.5}$$

In the rest of this section, we investigate differential operators of order 2 generated by (2.5), which maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$.

Theorem 2.2. *Suppose that* $\sigma(\xi,t)=\sum_{k=0}^m r_k(t)\xi^k$. *Then the operator* $\sigma(\mathcal{D},\mathcal{X})$ *is given by*

$$\sigma(\mathcal{D},\mathcal{X})=\sum_{k=0}^m\left(\frac{1}{2\pi\mathrm{i}}\right)^k\sum_{j=0}^k\binom{k}{j}\left(\frac{1}{2}\right)^{k-j}\frac{d^{k-j}}{dt^{k-j}}r_k(t)\frac{d^j}{dt^j}. \tag{2.6}$$

Proof. We write $\sigma(\mathcal{D},\mathcal{X})f(t)$ as

$$\begin{aligned}\sigma(\mathcal{D},\mathcal{X})f(t)&=\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma\left(\xi,\frac{t+y}{2}\right)\mathrm{e}^{2\pi\mathrm{i}(t-y)\xi}f(y)dyd\xi\\&=\sum_{k=0}^m\int_{\mathbb{R}}\int_{\mathbb{R}}r_k\left(\frac{t+y}{2}\right)\xi^k\mathrm{e}^{2\pi\mathrm{i}(t-y)\xi}f(y)dyd\xi\\&=\sum_{k=0}^m\int_{\mathbb{R}}\xi^k\left[\int_{\mathbb{R}}f(y)r_k\left(\frac{t+y}{2}\right)\mathrm{e}^{-2\pi\mathrm{i}y\xi}dy\right]\mathrm{e}^{2\pi\mathrm{i}t\xi}d\xi\\&=\sum_{k=0}^m\int_{\mathbb{R}}\xi^k\mathcal{F}\left[f(\cdot)r_k\left(\frac{t+\cdot}{2}\right)\right](\xi)\mathrm{e}^{2\pi\mathrm{i}t\xi}d\xi\\&=\sum_{k=0}^m\mathcal{F}^{-1}\left\{\xi^k\mathcal{F}\left[f(\cdot)r_k\left(\frac{t+\cdot}{2}\right)\right](\xi)\right\}(t).\end{aligned}$$

By using the relation $\mathcal{F}^{-1}(\cdot^k\mathcal{F}f(\cdot))(x)=(\frac{1}{2\pi\mathrm{i}})^k\frac{d^k}{dx^k}f(x)$, it follows that

$$\begin{aligned}\sigma(\mathcal{D},\mathcal{X})f(t)&=\sum_{k=0}^m\left(\frac{1}{2\pi\mathrm{i}}\right)^k\frac{d^k}{dx^k}\left[f(x)r_k\left(\frac{t+x}{2}\right)\right]_{x=t}\\&=\sum_{k=0}^m\left(\frac{1}{2\pi\mathrm{i}}\right)^k\sum_{j=0}^k\binom{k}{j}\left(\frac{1}{2}\right)^{k-j}\frac{d^{k-j}}{dt^{k-j}}r_k(t)\frac{d^j}{dt^j}f(t).\end{aligned}$$

This completes the proof. □

In particular, when $m = 2$, we have

$$
\begin{aligned}
&\sigma(\mathcal{D},\mathcal{X})f(t)\\
&= \sum_{k=0}^{2}\left(\frac{1}{2\pi \mathrm{i}}\right)^{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{2}\right)^{k-j}\frac{d^{k-j}}{dt^{k-j}}r_k(t)\frac{d^j}{dt^j}f(t)\\
&= r_0(t)f(t) + \frac{1}{2\pi \mathrm{i}}\left(\frac{1}{2}r_1'(t)f(t) + r_1(t)f'(t)\right) - \frac{1}{4\pi^2}\left(\frac{1}{4}r_2''(t)f(t) + r_2'(t)f'(t) + r_2(t)f''(t)\right)\\
&= \left(r_0(t) + \frac{1}{4\pi \mathrm{i}}r_1'(t) - \frac{1}{16\pi^2}r_2''(t)\right)f(t) + \left(\frac{1}{2\pi \mathrm{i}}r_1(t) - \frac{1}{4\pi^2}r_2'(t)\right)f'(t) - \frac{1}{4\pi^2}r_2(t)f''(t).
\end{aligned}
$$

Therefore, we obtain the following corollary.

Corollary 2.3. *Set* $\sigma(\xi,t) = r_0(t) + r_1(t)\xi + r_2(t)\xi^2$. *The operator* $\sigma(\mathcal{D},\mathcal{X})$ *defined in* (2.5) *is*

$$
\sigma(\mathcal{D},\mathcal{X}) = -\frac{r_2(t)}{4\pi^2}\frac{d^2}{dt^2} + \left(\frac{1}{2\pi \mathrm{i}}r_1(t) - \frac{1}{4\pi^2}r_2'(t)\right)\frac{d}{dt} + \left(r_0(t) + \frac{1}{4\pi \mathrm{i}}r_1'(t) - \frac{1}{16\pi^2}r_2''(t)\right)\mathcal{I}. \tag{2.7}
$$

Theorem 2.4. *Suppose that* $\sigma(\mathcal{D},\mathcal{X})$ *is defined b* (2.5). *Then* $\sigma(\mathcal{D},\mathcal{X}) = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t)\mathcal{I}$ *onl if*

$$
\sigma(\xi,t) = \left(c(t) - \frac{1}{2}b'(t) + \frac{1}{4}a''(t)\right) + 2\pi \mathrm{i}(b(t) - a'(t))\xi - 4\pi^2 a(t)\xi^2. \tag{2.8}
$$

Proof. Set $\sigma(\xi,t) = r_0(t) + r_1(t)\xi + r_2(t)\xi^2$. Equation (2.7) leads to a relationship between (a,b,c) and (r_0,r_1,r_2) as follows:

$$
\begin{aligned}
a(t) &= -\frac{1}{4\pi^2}r_2(t),\\
b(t) &= \frac{1}{2\pi \mathrm{i}}r_1(t) - \frac{1}{4\pi^2}r_2'(t),\\
c(t) &= r_0(t) + \frac{1}{4\pi \mathrm{i}}r_1'(t) - \frac{1}{16\pi^2}r_2''(t).
\end{aligned}
$$

Solving these equations, we obtain that $r_2(t) = -4\pi^2 a(t)$, $r_1(t) = 2\pi \mathrm{i}[b(t) + \frac{1}{4\pi^2}r_2'(t)] = 2\pi \mathrm{i}[b(t) - a'(t)]$, and

$$
\begin{aligned}
r_0(t) &= c(t) - \frac{1}{4\pi \mathrm{i}}r_1'(t) + \frac{1}{16\pi^2}r_2''(t)\\
&= c(t) - \frac{1}{4\pi \mathrm{i}}2\pi \mathrm{i}[b(t) - a'(t)]' + \frac{1}{16\pi^2}[-4\pi^2 a(t)]''\\
&= c(t) - \frac{1}{2}b'(t) + \frac{1}{4}a''(t).
\end{aligned}
$$

This completes the proof of this theorem. □

We now turn to the Sturm-Liouville operator. A general Sturm-Liouville operator $\mathcal{L}_{u,v}$ has the form $\mathcal{L}_{u,v} = u(t)\frac{d}{dt}v(t)\frac{d}{dt}$, where u and v are de ned on an interval $I = (a,b), (a,+\infty)$, or $\mathbb{R}$.

Theorem 2.5. *The Sturm-Liou ille operator* $\mathcal{L}_{u,v}$ *can be generated through* (2.5) *ith the kernel*

$$
\sigma(\xi,t) = \frac{1}{4}[u''(t)v(t) - u(t)v''(t)] - 2\pi \mathrm{i}u'(t)v(t)\xi - 4\pi^2 u(t)v(t)\xi^2. \tag{2.9}
$$

Proof. The identity (2.9) is a direct consequence of (2.8) for $a(t) = u(t)v(t)$, $b(t) = u(t)v'(t)$ and $c(t) = 0$. □

Corollary 2.6. *The kernel function* $\sigma(\xi,t) = -2\pi \mathrm{i}u(t)u'(t)\xi - 4\pi^2u^2(t)\xi^2$ *determines the operator*

$$
\mathcal{L}_u = u(t)\frac{d}{dt}u(t)\frac{d}{dt} = u^2(t)\frac{d^2}{dt^2} + u(t)u'(t)\frac{d}{dt}. \tag{2.10}
$$

Throughout the paper, we will investigate the generalized Sturm-Liouville operators

$$\mathcal{L} = \mathcal{L}_u + \wp_{\alpha,\beta} \tag{2.11}$$

with $\wp_{\alpha,\beta} = \alpha(t)\frac{d}{dt} + \beta(t)\mathcal{I}$ for some di erential functions u, α and β. For the kernel of the operator $\mathcal{L}$, we have the following theorem.

Theorem 2.7. *The kernel function*

$$\sigma(\xi,t) = \left(\beta(t) - \frac{1}{2}\alpha'(t)\right) + (2\pi\mathrm{i}\alpha(t) - 2\pi\mathrm{i}u(t)u'(t))\xi - 4\pi^2u^2(t)\xi^2$$

determines the operator $\mathcal{L}$ *defined in* (2.11).

Proof. By Theorem 2.4, we know that $\sigma(\mathcal{D},\mathcal{X}) = \alpha(t)\frac{d}{dt} + \beta(t)\mathcal{I}$ if and only if $\sigma(\xi,t) = (\beta(t) - \frac{1}{2}\alpha'(t)) + 2\pi\mathrm{i}\alpha(t)\xi$. Combining this with Corollary 2.6 and (2.11), the proof is completed. □

3 Spectral operators of the Laguerre systems and the Kautz systems

For any parameter a in the unit disc, $\mathbb{D} := \{z = x + \mathrm{i}y : |z| < 1\}$. Denote the Poisson kernel for the unit disc by

$$p_a(t) = \frac{1-|a|^2}{|1-a\mathrm{e}^{\mathrm{i}t}|^2}, \quad t \in \mathbb{T}. \tag{3.1}$$

De ne the real-valued function $\theta_a : \mathbb{T} = (-\pi,\pi) \to \mathbb{R}$ by $\mathrm{e}^{\mathrm{i}\theta_a(t)} = \frac{z-a}{1-\bar{a}z}|_{z=\mathrm{e}^{\mathrm{i}t}}$, $t \in \mathbb{T}$ with the extension principle $\theta_a(t+2\pi) = 2\pi + \theta_a(t)$. We note that the density function of the harmonic measure is $\theta_a'(t) = p_a(t)$.

Suppose that $\mathcal{L}_u$ is de ned in (2.10) with $u = \frac{1}{p_a(\cdot)}$. De ne the di erential operator $\mathcal{L}$ by

$$\mathcal{L} = \mathcal{L}_u + \wp_u \tag{3.2}$$

with $\wp_u = \wp_{\alpha,\beta}$ and $\alpha(t) = \frac{2\mathrm{i}}{(1-a\mathrm{e}^{-\mathrm{i}t})p_a^2(t)}$ and $\beta(t) = \frac{\mathrm{i}}{(1-a\mathrm{e}^{-\mathrm{i}t})p_a^2(t)}(\mathrm{i} - \frac{p_a'(t)}{p_a(t)})$.

The main result of this section is the following.

Theorem 3.1. *Suppose that* $\mathcal{L}$ *is defined in* (3.2). *Then*

$$\mathcal{L}(e_n(t)) = -(n+1)^2e_n(t). \tag{3.3}$$

Proof. To proceed with the proof, we need two identities. The rst is

$$\frac{d}{dt}e_n(t) = \mathrm{i}\left((n+1)p_a(t) - \frac{1}{1-a\mathrm{e}^{-\mathrm{i}t}}\right)e_n(t), \quad t \in \mathbb{T}.$$

The second is

$$\frac{d^2}{dt^2}e_n(t) = \mathrm{i}(n+1)p_a'(t)e_n(t) - \frac{a\mathrm{e}^{-\mathrm{i}t}}{(1-a\mathrm{e}^{-\mathrm{i}t})^2}e_n(t) - (n+1)^2p_a^2(t)e_n(t) - \frac{1}{(1-a\mathrm{e}^{-\mathrm{i}t})^2}e_n(t) + \frac{2(n+1)p_a(t)}{1-a\mathrm{e}^{-\mathrm{i}t}}e_n(t).$$

Using the representations of $\frac{d}{dt}e_n(t)$ and $\frac{d^2}{dt^2}e_n(t)$, we have

$$\frac{e_n''(t)}{p_a^2(t)} = \mathrm{i}(n+1)\frac{p_a'(t)}{p_a^2(t)}e_n(t) - \frac{a\mathrm{e}^{-\mathrm{i}t}}{(1-a\mathrm{e}^{-\mathrm{i}t})^2p_a^2(t)}e_n(t) - (n+1)^2e_n(t) - \frac{1}{(1-a\mathrm{e}^{-\mathrm{i}t})^2p_a^2(t)}e_n(t) + \frac{2(n+1)}{(1-a\mathrm{e}^{-\mathrm{i}t})p_a(t)}e_n(t)$$

and

$$-\frac{p_a'(t)}{p_a^3(t)}e_n'(t) = -\mathrm{i}\frac{p_a'(t)}{p_a^3(t)}\left((n+1)p_a(t) - \frac{1}{1-a\mathrm{e}^{-\mathrm{i}t}}\right)e_n(t)$$
$$= -\mathrm{i}\frac{p_a'(t)}{p_a^2(t)}(n+1)e_n(t) + \mathrm{i}\frac{p_a'(t)}{p_a^3(t)}\frac{1}{1-a\mathrm{e}^{-\mathrm{i}t}}e_n(t).$$

On one hand, this gives

$$\mathcal{L}_u(e_n(t)) = u^2(t)\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}t),$$

4.1 Boundary TM systems

Set

$$p_{\vec{a}_n}(t) = \sum_{j=0}^{n} p_{a_j}(t), \quad t \in \mathbb{T}, \tag{4.1}$$

where p_a is defined in (3.1).

Now recall the Sturm-Liouville operator $\mathcal{L}_u$ in (2.10). For the sequence $\{\vec{a}_n : n = 0, 1, \ldots\}$ of parameter vectors in the TM system, define the operator sequence $\mathcal{L}_{u_n,\mathbb{T}}$ by

$$\mathcal{L}_{u_n,\mathbb{T}} = u_n(t)\frac{d}{dt}u_n(t)\frac{d}{dt} = u_n^2(t)\frac{d^2}{dt^2} + u_n(t)u_n'(t)\, d$$

$$
\begin{aligned}
&= \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) - \frac{a_n\mathrm{e}^{-\mathrm{i}t}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) - e_{\vec{a}_n}(t)\\
&\quad - \frac{1}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\quad - \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t)\\
&= -\frac{a_n\mathrm{e}^{-\mathrm{i}t}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) - e_{\vec{a}_n}(t) - \frac{1}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\quad + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t)\\
&= -e_{\vec{a}_n}(t) + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) - \frac{1+a_n\mathrm{e}^{-\mathrm{i}t}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\quad + \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t).
\end{aligned}
$$

Conversely, we have

$$
\begin{aligned}
\frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\frac{d}{dt}e_{\vec{a}_n}(t) &= \frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\mathrm{i}\left(p_{\vec{a}_n}(t) - \frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}\right)e_{\vec{a}_n}(t)\\
&= -\frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t).
\end{aligned}
$$

The summation of $\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t))$ and $\frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\frac{d}{dt}e_{\vec{a}_n}(t)$ is

$$
\begin{aligned}
&\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) + \frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\frac{d}{dt}e_{\vec{a}_n}(t)\\
&\quad= -e_{\vec{a}_n}(t) + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) - \frac{1+a_n\mathrm{e}^{-\mathrm{i}t}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\qquad + \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t) - \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \frac{2}{(1-a_n\mathrm{e}^{-\mathrm{i}t})^2p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\quad= -e_{\vec{a}_n}(t) + \frac{1}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t) + \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) + \frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\frac{d}{dt}e_{\vec{a}_n}(t) - \frac{1}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\\
&\quad - \mathrm{i}\frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}\frac{1}{1-a_n\mathrm{e}^{-\mathrm{i}t}}e_{\vec{a}_n}(t) = -e_{\vec{a}_n}(t),
\end{aligned}
$$

from which, we obtain

$$
\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) + \frac{2\mathrm{i}}{(1-a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\left(\frac{d}{dt} + \frac{\mathrm{i}}{2} - \frac{1}{2}\frac{p'_{\vec{a}_n}(t)}{p_{\vec{a}_n}(t)}\right)e_{\vec{a}_n} = -e_{\vec{a}_n}.
$$

By noting the de nition of $\wp_{u_n,\mathbb{T}}$ in (4.3), we see that the above equation is equivalent to

$$
\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) + \wp_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) = -e_{\vec{a}_n}(t).
$$

This con rms (4.5) and nishes the proof of the theorem. □

Remark 4.2. Equation (4.5) does not reduce to (3.3) when all the parameters a_j's are identical. In fact, when all a_j's are equal to a, the operators $\mathcal{L}_{u_n,\mathbb{T}}$ and $\wp_{u_n,\mathbb{T}}$, respectively, reduce to $\mathcal{L}_u$ and $\wp$ up to the factor $\frac{1}{n^2}$, i.e., $\mathcal{L}_{u_n,\mathbb{T}} = \frac{1}{n^2}\mathcal{L}_u$ and $\wp_{u_n,\mathbb{T}} = \frac{1}{n^2}\wp_u$. It is clear that (4.5) gives rise to (3.3).

4.2 Complex variable setting

For $a \in \mathbb{D}$, denoted by $P_a(z)$ the Poisson kernel for the unit disc can be written as follows:

$$P_a(z) = \frac{1-|a|^2}{(z-a)(1-az)}, \quad z \in \mathbb{D}. \tag{4.6}$$

Set

$$P_{\vec{a}_n}(z) = \sum_{j=0}^{n} P_{a_j}(z), \quad z \in \mathbb{D} \tag{4.7}$$

for vector $\vec{a}_n = (a_0, \ldots, a_{n-1}, a_n)^{\mathrm{T}} \in \mathbb{D}^{n+1}$. De ne $\mathcal{L}_{u_n,\mathbb{D}}$ by

$$\mathcal{L}_{u_n,\mathbb{D}} = u_n(z)\frac{d}{dz}u_n(z)\frac{d}{dz} = \frac{1}{P^2_{\vec{a}_n}(z)}\frac{d^2}{dz^2} - \frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}\frac{d}{dz} \tag{4.8}$$

with $u_n = \frac{1}{P_{\vec{a}_n}}$. Let

$$\wp_{u_n,\mathbb{D}} = \frac{2}{(z-a_n)P^2_{\vec{a}_n}(z)}\left(\frac{d}{dz} - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}\right) \tag{4.9}$$

and

$$\mathcal{L}_{n,\mathbb{D}} = \mathcal{L}_{u_n,\mathbb{D}} + \wp_{u_n,\mathbb{D}}. \tag{4.10}$$

Theorem 4.3. *Suppose that the operator* $\mathcal{L}_{n,\mathbb{D}}$ *and the s stem* $\{E_{\vec{a}_n}\}$ *are defined in* (4.10) *and* (1.1), *respecti el . Then*

$$\mathcal{L}_{n,\mathbb{D}}(E_{\vec{a}_n}(z)) = E_{\vec{a}_n}(z), \quad z \in \mathbb{D}. \tag{4.11}$$

Proof. With disintegrated computation, the proof is like that of Theorem 4.1 by invoking the identity

$$\frac{d}{dz}E_{\vec{a}_n}(z) = \left(P_{\vec{a}_{n-1}}(z) + \frac{a_n}{1-a_n z}\right)E_{\vec{a}_n}(z) = \left(P_{\vec{a}_n}(z) - \frac{1}{z-a_n}\right)E_{\vec{a}_n}(z), \quad z \in \mathbb{D}.$$

This completes the proof. □

4.3 From the complex TM system to the boundary TM system

This section addresses the relationship between $\mathcal{L}_{n,\mathbb{D}}$ and $\mathcal{L}_{n,\mathbb{T}}$. We will prove that by setting $z = \mathrm{e}^{\mathrm{i}t}$ in (4.11) so that we obtain (4.5). In fact, by setting

$$\begin{aligned} R(t) = {} & \frac{1}{P^2_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}E''_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) - \frac{P'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}{P^3_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}E'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) \\ & + \frac{2}{(\mathrm{e}^{\mathrm{i}t}-a_n)P^2_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}\left[E'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) - \frac{1}{2}\frac{P'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}{P_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})}E_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t})\right], \end{aligned}$$

and $L(t) = E_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) = e_{\vec{a}_n}(t)$, $t \in \mathbb{T}$, and by setting $z = \mathrm{e}^{\mathrm{i}t}$ in (4.11), the relation can be written as $R(t) = L(t)$.

The following theorem indicates that $R(t) = -\mathcal{L}_{n,\mathbb{T}}(e_{\vec{a}_n}(t))$, from which we conclude that (4.11) with $z = \mathrm{e}^{\mathrm{i}t}$ leads to (4.5).

Theorem 4.4. *Suppose that* $\mathcal{L}_{n,\mathbb{T}}$ *is the operator defined in* (4.4) *and* $R(t)$ *is the right-hand side of* (4.11) *ith* $z = \mathrm{e}^{\mathrm{i}t}$. *Then* $R(t) = -\mathcal{L}_{n,\mathbb{T}}(e_{\vec{a}_n}(t))$.

Proof. Using the relations between $E_{\vec{a}_n}$ and $e_{\vec{a}_n}$,

$$\begin{aligned} E'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) &= -\mathrm{i}\mathrm{e}^{-\mathrm{i}t}e'_{\vec{a}_n}(t), \quad t \in \mathbb{T}, \\ E''_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) &= \mathrm{i}\mathrm{e}^{-2\mathrm{i}t}(e'_{\vec{a}_n}(t) + \mathrm{i}e''_{\vec{a}_n}(t)), \quad t \in \mathbb{T}, \end{aligned}$$

and similar relations for $P_{\vec{a}_n}$ and $p_{\vec{a}_n}$,

$$P_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{-\mathrm{i}t}p_{\vec{a}_n}(t), \quad t \in \mathbb{T},$$

$$P'_{\vec{a}_n}(\mathrm{e}^{\mathrm{i}t}) = -\mathrm{e}^{-2\mathrm{i}t}(p_{\vec{a}_n}(t) + \mathrm{i}p'_{\vec{a}_n}(t)), \quad t \in \mathbb{T},$$

we have

$$\begin{aligned} R(t) &= \frac{1}{\mathrm{e}^{-2\mathrm{i}t}p^2_{\vec{a}_n}(t)}\mathrm{i}\mathrm{e}^{-2\mathrm{i}t}(e'_{\vec{a}_n}(t) + \mathrm{i}e''_{\vec{a}_n}(t)) - \frac{-\mathrm{e}^{-2\mathrm{i}t}(p_{\vec{a}_n}(t) + \mathrm{i}p'_{\vec{a}_n}(t))}{\mathrm{e}^{-3\mathrm{i}t}p^3_{\vec{a}_n}(t)}(-\mathrm{i})\mathrm{e}^{-\mathrm{i}t}e'_{\vec{a}_n}(t) \\ &\quad + \frac{2}{(\mathrm{e}^{\mathrm{i}t} - a_n)\mathrm{e}^{-2\mathrm{i}t}p^2_{\vec{a}_n}(t)}\left[-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}e'_{\vec{a}_n}(t) - \frac{1}{2}\frac{-\mathrm{e}^{-2\mathrm{i}t}(p_{\vec{a}_n}(t) + \mathrm{i}p'_{\vec{a}_n}(t))}{\mathrm{e}^{-\mathrm{i}t}p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\right] \\ &= \frac{1}{p^2_{\vec{a}_n}(t)}(\mathrm{i}e'_{\vec{a}_n}(t) - e''_{\vec{a}_n}(t)) - \frac{\mathrm{i}(p_{\vec{a}_n}(t) + \mathrm{i}p'_{\vec{a}_n}(t))}{p^3_{\vec{a}_n}(t)}e'_{\vec{a}_n}(t) \\ &\quad + \frac{2}{(\mathrm{e}^{\mathrm{i}t} - a_n)\mathrm{e}^{-\mathrm{i}t}p^2_{\vec{a}_n}(t)}\left[-\mathrm{i}e'_{\vec{a}_n}(t) + \frac{1}{2}\frac{(p_{\vec{a}_n}(t) + \mathrm{i}p'_{\vec{a}_n}(t))}{p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\right] \\ &= -\frac{e''_{\vec{a}_n}(t)}{p^2_{\vec{a}_n}(t)} + \frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}e'_{\vec{a}_n}(t) \\ &\quad + \frac{2}{(1 - a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\left[-\mathrm{i}e'_{\vec{a}_n}(t) + \frac{1}{2}e_{\vec{a}_n}(t) + \frac{\mathrm{i}}{2}\frac{p'_{\vec{a}_n}(t)}{p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\right] \\ &= -\frac{e''_{\vec{a}_n}(t)}{p^2_{\vec{a}_n}(t)} + \frac{p'_{\vec{a}_n}(t)}{p^3_{\vec{a}_n}(t)}e'_{\vec{a}_n}(t) \\ &\quad + \frac{2\mathrm{i}}{(1 - a_n\mathrm{e}^{-\mathrm{i}t})p^2_{\vec{a}_n}(t)}\left[-e'_{\vec{a}_n}(t) - \frac{\mathrm{i}}{2}e_{\vec{a}_n}(t) + \frac{1}{2}\frac{p'_{\vec{a}_n}(t)}{p_{\vec{a}_n}(t)}e_{\vec{a}_n}(t)\right] \\ &= -\mathcal{L}_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) - \wp_{u_n,\mathbb{T}}(e_{\vec{a}_n}(t)) = -\mathcal{L}_{n,\mathbb{T}}(e_{\vec{a}_n}(t)). \end{aligned}$$

This completes the proof of the theorem. □

5 Upper half-plane

There is a parallel theory for TM systems on the upper half-plane. In the upper half-plane context, we say $f \in H^p(\mathbb{C}^+)$, $0 < p < \infty$, if f is analytic on $\mathbb{C}^+$ and $\sup_{y>0} \int_{-\infty}^{\infty} |f(x + \mathrm{i}y)|^p dx = \|f\|^p_{H^p(\mathbb{C}^+)} < \infty$. When $p = \infty$, we write $f \in H^\infty(\mathbb{C}^+)$ for the totality of all the bounded analytic functions on $\mathbb{C}^+$, and we give $H^\infty(\mathbb{C}^+)$ the norm $\|f\|_{H^\infty(\mathbb{C}^+)} = \sup_{w\in\mathbb{C}^+} |f(w)|$. The relation between $f \in H^p(\mathbb{C}^+)$ and their non-tangential boundary limits on $\mathbb{R}$ is the same as for the unit disc. $H^2(\mathbb{C}^+)$ has inner product

$$\langle f, g\rangle_{\mathbb{C}^+} = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt, \quad f, g \in H^2(\mathbb{C}^+).$$

For a given parameter sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}^+$, the corresponding TM system $\{\beta_{\vec{\lambda}_n}\}_{n=0}^{\infty}$ on the upper half-plane $\mathbb{C}^+$ is

$$\beta_{\vec{\lambda}_n}(z) = \frac{\sqrt{\frac{1}{\pi}\mathrm{Im}\{\lambda_n\}}}{z - \overline{\lambda}_n}\prod_{j=0}^{n-1}\frac{z - \lambda_j}{z - \overline{\lambda}_j}, \quad z \in \mathbb{C}^+, \quad n \in \mathbb{Z}_+, \tag{5.1}$$

with vector $\vec{\lambda}_n = (\lambda_0, \dots, \lambda_{n-1}, \lambda_n)^{\mathrm{T}} \in (\mathbb{C}^+)^{n+1}$. Under the condition $\sum_{k=0}^{\infty} \frac{\sqrt{\mathrm{Im}(\lambda_k)}}{1+|\lambda_k|^2} = \infty$, $\{\beta_{\vec{\lambda}_n}\}$ is an orthonormal basis of the Hardy space $H^p(\mathbb{C}^+), 1 < p < \infty$.

The corresponding boundary TM system $\{\beta_{\vec{\lambda}_n}\}_{n=0}^{\infty}$ on the upper-half plane case is

$$\beta_{\vec{\lambda}_n}(t) := \lim_{\mathrm{Im}(z)\to 0^+} \beta_{\vec{\lambda}_n}(z) = \frac{\sqrt{\frac{1}{\pi}\mathrm{Im}\{\lambda_n\}}}{t - \overline{\lambda}_n}\prod_{j=0}^{n-1}\frac{t - \lambda_j}{t - \overline{\lambda}_j}, \quad t \in \mathbb{R}, \quad n = 0, 1, \dots \tag{5.2}$$

5.1 For a boundary TM system on the upper half-plane

For $\lambda \in \mathbb{C}^+$, the Poisson kernel q_λ on the real line is de ned by

$$q_\lambda(t) = \frac{2\mathrm{Im}\{\lambda\}}{|t-\lambda|^2}, \quad t \in \mathbb{R}. \tag{5.3}$$

Generally, for vector $\vec{\lambda}_n = (\lambda_0, \ldots, \lambda_{n-1}, \lambda_n)^{\mathrm{T}}$, set

$$q_{\vec{\lambda}_n}(t) = \sum_{j=0}^{n} q_{\lambda_j}(t), \quad t \in \mathbb{R}. \tag{5.4}$$

For a sequence $\{\vec{\lambda}_n : n = 0, 1, \ldots\}$ of parameter vectors on the upper half-plane, de ne the operator sequence $\mathcal{L}_{u_n}$ by

$$\mathcal{L}_{u_n,\mathbb{R}} = u_n(t)\frac{d}{dt}u_n(t)\frac{d}{dt} = u_n^2(t)\frac{d^2}{dt^2} + u_n(t)u_n'(t)\frac{d}{dt} \tag{5.5}$$

with $u_n = \frac{1}{q_{\vec{\lambda}_n}}$. Essentially,

$$\mathcal{L}_{u_n,\mathbb{R}} = \frac{1}{q^2_{\vec{\lambda}_n}(t)}\frac{d^2}{dt^2} - \frac{q'_{\vec{\lambda}_n}(t)}{q^3_{\vec{\lambda}_n}(t)}\frac{d}{dt}.$$

Set

$$\wp_{u_n,\mathbb{R}} = \frac{2}{(t-\lambda_n)q^2_{\vec{\lambda}_n}(t)}\left(\frac{d}{dt} - \frac{1}{2}\frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)}\right) \tag{5.6}$$

and

$$\mathcal{L}_{n,\mathbb{R}} = \mathcal{L}_{u_n,\mathbb{R}} + \wp_{u_n,\mathbb{R}}. \tag{5.7}$$

Theorem 5.1. *Suppose that the operator* $\mathcal{L}_{n,\mathbb{R}}$ *and the s stem* $\{\beta_{\vec{\lambda}_n}\}$ *are defined in* (5.7) *and* (5.2)*, respecti el . Then*

$$\mathcal{L}_{n,\mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) = -\beta_{\vec{\lambda}_n}(t), \quad t \in \mathbb{R}. \tag{5.8}$$

Proof. The identity

$$(t-\lambda_n)\beta_{\vec{\lambda}_n}(t) = \sqrt{\frac{1}{\pi}\mathrm{Im}(\lambda_n)}\prod_{j=0}^{n}\frac{t-\lambda_j}{t-\overline{\lambda}_j}$$

implies that

$$\beta'_{\vec{\lambda}_n}(t) = \left(\mathrm{i}q_{\vec{\lambda}_n}(t) - \frac{1}{t-\lambda_n}\right)\beta_{\vec{\lambda}_n}(t).$$

Furthermore, we have

$$\beta''_{\vec{\lambda}_n}(t) = \left[\mathrm{i}q'_{\vec{\lambda}_n}(t) + \frac{2}{(t-\lambda_n)^2} - q^2_{\vec{\lambda}_n}(t) - \frac{2\mathrm{i}}{t-\lambda_n}q_{\vec{\lambda}_n}(t)\right]\beta_{\vec{\lambda}_n}(t).$$

Then

$$\begin{aligned}
\mathcal{L}_{u_n,\mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) &= \frac{1}{q^2_{\vec{\lambda}_n}(t)}\beta''_{\vec{\lambda}_n}(t) - \frac{q'_{\vec{\lambda}_n}(t)}{q^3_{\vec{\lambda}_n}(t)}\beta'_{\vec{\lambda}_n}(t) \\
&= \frac{1}{q^2_{\vec{\lambda}_n}(t)}\left(\mathrm{i}q'_{\vec{\lambda}_n}(t) + \frac{2}{(t-\lambda_n)^2} - q^2_{\vec{\lambda}_n}(t) - \frac{2\mathrm{i}}{t-\lambda_n}q_{\vec{\lambda}_n}(t)\right)\beta_{\vec{\lambda}_n}(t) \\
&\quad - \frac{q'_{\vec{\lambda}_n}(t)}{q^3_{\vec{\lambda}_n}(t)}\left(\mathrm{i}q_{\vec{\lambda}_n}(t) - \frac{1}{t-\lambda_n}\right)\beta_{\vec{\lambda}_n}(t) \\
&= \left(\frac{2}{(t-\lambda_n)^2 q^2_{\vec{\lambda}_n}(t)} - 1 - \frac{2\mathrm{i}}{(t-\lambda_n)q_{\vec{\lambda}_n}(t)} + \frac{q'_{\vec{\lambda}_n}(t)}{(t-\lambda_n)q^3_{\vec{\lambda}_n}(t)}\right)\beta_{\vec{\lambda}_n}(t)
\end{aligned}$$

and

$$
\begin{aligned}
\wp_{u_n,\mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) &= \frac{2}{(t-\lambda_n)q^2_{\vec{\lambda}_n}(t)}\left(\beta'_{\vec{\lambda}_n}(t) - \frac{1}{2}\frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)}\beta_{\vec{\lambda}_n}(t)\right)\\
&= \frac{2}{(t-\lambda_n)q^2_{\vec{\lambda}_n}(t)}\left(\left(\mathrm{i}q_{\vec{\lambda}_n}(t) - \frac{1}{t-\lambda_n}\right)\beta_{\vec{\lambda}_n}(t) - \frac{1}{2}\frac{q'_{\vec{\lambda}_n}(t)}{q_{\vec{\lambda}_n}(t)}\beta_{\vec{\lambda}_n}(t)\right)\\
&= \left(\frac{2\mathrm{i}}{(t-\lambda_n)q_{\vec{\lambda}_n}(t)} - \frac{2}{(t-\lambda_n)^2q^2_{\vec{\lambda}_n}(t)} - \frac{q'_{\vec{\lambda}_n}(t)}{(t-\lambda_n)q^3_{\vec{\lambda}_n}(t)}\right)\beta_{\vec{\lambda}_n}(t).
\end{aligned}
$$

Therefore, $\mathcal{L}_{u_n,\mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) + \wp_{u_n,\mathbb{R}}(\beta_{\vec{\lambda}_n}(t)) = -\beta_{\vec{\lambda}_n}(t)$. This completes the proof of this theorem. □

5.2 For TM systems on the upper half-plane

For $\lambda \in \mathbb{C}^+$, we extend the Poisson kernel q_λ to the complex plane

$$Q_\lambda(z) = \frac{2\mathrm{Im}\{\lambda\}}{(z-\lambda)(z-\lambda)}, \quad z \in \mathbb{C} \tag{5.9}$$

and

$$Q_{\vec{\lambda}_n}(z) = \sum_{j=0}^{n} Q_{\lambda_j}(z), \quad z \in \mathbb{C} \tag{5.10}$$

for vector $\vec{\lambda}_n = (\lambda_0, \ldots, \lambda_{n-1}, \lambda_n)^{\mathrm{T}} \in (\mathbb{C}^+)^{n+1}$.

For the sequence $\{\vec{\lambda}_n : n = 0, 1, \ldots\}$ of parameter vectors, define the operator sequence $\mathcal{L}_{u_n,\mathbb{C}^+}$ by

$$\mathcal{L}_{u_n,\mathbb{C}^+} = u_n(z)\frac{d}{dz}u_n(z)\frac{d}{dz} = \frac{1}{\mathbf{Q}^2_{\vec{\lambda}_n}(z)}\ d^2$$

6 From the upper half-plane to the unit disc

This section will establish a relationship between the operators $\mathcal{L}_{n,\mathbb{C}^+}$ and $\mathcal{L}_{n,\mathbb{D}}$. To build a bridge between $\mathcal{L}_{n,\mathbb{C}^+}$ and $\mathcal{L}_{n,\mathbb{D}}$, we use the Caley transform $\kappa : \mathbb{D} \to \mathbb{C}^+$:

$$\kappa(z) = \mathrm{i}\frac{1-z}{1+z}, \quad z \in \mathbb{D},$$

which is a bijection between $\mathbb{D}$ and $\mathbb{C}^+$ with inverse $\kappa^{-1} : \mathbb{C}^+ \to \mathbb{D}$,

$$\kappa^{-1}(z) = \frac{\mathrm{i}-z}{\mathrm{i}+z}, \quad z \in \mathbb{C}^+.$$

The following identity is crucial. It establishes a relationship between $\beta_{\vec{\lambda}_n}$ and $E_{\vec{a}_n}$ through the composition of κ:

$$\beta_{\vec{\lambda}_n}(\kappa(z)) = C_1(1+z)E_{\vec{a}_n}(z), \quad z \in \mathbb{D},$$

where $(\lambda_j, a_j) \in \mathbb{C}^+ \times \mathbb{D}$ is the κ-pair ruled by $\lambda_j = \kappa(a_j) = \mathrm{i}\frac{1-a_j}{1+a_j}$ or $a_j = \kappa^{-1}(\lambda_j) = \frac{\mathrm{i}-\lambda_j}{\mathrm{i}+\lambda_j}$ for $j \in \{0, 1, \ldots, n\}$, $C_1 = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{\mathrm{i}\alpha}$, and

$$\mathrm{e}^{\mathrm{i}\alpha} = (-1)^{n+1}\mathrm{i}\frac{1+a_n}{|1+a_n|}\overset{n-1}{\underset{j=0}{}}\frac{1+a_j}{1+a_j}.$$

Replacing $z \in \mathbb{C}^+$ by $\kappa(z) \in \mathbb{C}^+, z \in \mathbb{D}$, in (5.14), we get $R(z) = L(z)$, $z \in \mathbb{D}$, with

$$\begin{aligned} R(z) = {} & \frac{1}{Q^2_{\vec{\lambda}_n}(\kappa(z))}\beta''_{\vec{\lambda}_n}(\kappa(z)) - \frac{Q'_{\vec{\lambda}_n}(\kappa(z))}{Q^3_{\vec{\lambda}_n}(\kappa(z))}\beta'_{\vec{\lambda}_n}(\kappa(z)) \\ & + \frac{2}{(\kappa(z)-\lambda_n)Q^2_{\vec{\lambda}_n}(\kappa(z))}\left[\beta'_{\vec{\lambda}_n}(\kappa(z)) - \frac{1}{2}\frac{Q'_{\vec{\lambda}_n}(\kappa(z))}{Q_{\vec{\lambda}_n}(\kappa(z))}\beta_{\vec{\lambda}_n}(\kappa(z))\right] \end{aligned}$$

and

$$L(z) = -\beta_{\vec{\lambda}_n}(\kappa(z)) = -C_1(1+z)E_{\vec{a}_n}(z), \quad z \in \mathbb{D}.$$

The following theorem shows that $R(z) = -C_1(1+z)\mathcal{L}_{n,\mathbb{D}}(E_{\vec{a}_n}(z))$ so that (5.14) reduces to (4.11) when we replace $z \in \mathbb{C}^+$ by $\kappa(z)$ for $z \in \mathbb{D}$ in (5.14).

Theorem 6.1. *Suppose that $\mathcal{L}_{n,\mathbb{D}}$ is the operator defined in* (4.10) *and $R(z)$ is the right-hand side of* (5.14) *hen e replace $z \in \mathbb{C}^+$ b $\kappa(z), z \in \mathbb{D}$. Then $R(z) = -C_1(1+z)\mathcal{L}_{n,\mathbb{D}}(E_{\vec{a}_n}(z))$, $z \in \mathbb{D}$.*

Proof. We rst prove two identities associated with the derivatives of $\beta_{\vec{\lambda}_n}$:

$$\beta'_{\vec{\lambda}_n}(\kappa(z)) = \frac{\mathrm{i}}{2}C_1(1+z)^2[E_{\vec{a}_n}(z) + (1+z)E'_{\vec{a}_n}(z)], \quad z \in \mathbb{D}$$

and

$$\beta''_{\vec{\lambda}_n}(\kappa(z)) = -C_1\left[\frac{1}{4}(1+z)^5E''_{\vec{a}_n}(z) + (1+z)^4E'_{\vec{a}_n}(z) + \frac{1}{2}(1+z)^3E_{\vec{a}_n}(z)\right], \quad z \in \mathbb{D}.$$

Similarly, by noting that $\frac{2\mathrm{Im}(\lambda)}{|1-\mathrm{i}\lambda|^2} = \frac{1}{2}(1-|a|^2)$ for κ-pair (λ, a), $Q_{\vec{\lambda}_n}$ satis es both

$$Q_{\vec{\lambda}_n}(\kappa(z)) = \frac{1}{2}(1+z)^2P_{\vec{a}_n}(z), \quad z \in \mathbb{D},$$

and

$$Q'_{\vec{\lambda}_n}(\kappa(z)) = \frac{\mathrm{i}}{2}(1+z)^3P_{\vec{a}_n}(z) + \frac{\mathrm{i}}{4}(1+z)^4P'_{\vec{a}_n}(z), \quad z \in \mathbb{D}.$$

By substituting the expressions of $Q_{\vec{\lambda}_n}(\kappa(z))$, $Q'_{\vec{\lambda}_n}(\kappa(z))$ and $\beta^{(j)}_{\vec{\lambda}_n}(\kappa(z)), j = 0,1,2$, into the above equation, it follows that

$$\begin{aligned} R(z) = {}& 4(1+z)^{-4}\frac{1}{P^2_{\vec{a}_n}(z)}(-C_1)\left[\frac{1}{4}(1+z)^5E''_{\vec{a}_n}(z) + (1+z)^4E'_{\vec{a}_n}(z) + \frac{1}{2}(1+z)^3E_{\vec{a}_n}(z)\right] \\ & - \frac{\frac{\mathrm{i}}{2}(1+z)^3P_{\vec{a}_n}(z) + \frac{\mathrm{i}}{4}(1+z)^4P'_{\vec{a}_n}(z)}{\frac{1}{8}(1+z)^6P^3_{\vec{a}_n}(z)}\frac{\mathrm{i}}{2}C_1(1+z)^2[E_{\vec{a}_n}(z) + (1+z)E'_{\vec{a}_n}(z)] \\ & + \frac{2}{(\kappa(z)-\lambda_n)\frac{1}{4}(1+z)^4P^2_{\vec{a}_n}(z)}\Bigg\{\frac{\mathrm{i}}{2}C_1(1+z)^2[E_{\vec{a}_n}(z) + (1+z)E'_{\vec{a}_n}(z)] \\ & - \frac{1}{2}\frac{\frac{\mathrm{i}}{2}(1+z)^3P_{\vec{a}_n}(z) + \frac{\mathrm{i}}{4}(1+z)^4P'_{\vec{a}_n}(z)}{\frac{1}{2}(1+z)^2P_{\vec{a}_n}(z)}C_1(1+z)E_{\vec{a}_n}(z)\Bigg\}. \end{aligned}$$

After some calculations, we get

$$\begin{aligned} -\frac{1}{C_1(1+z)}R(z) = {}& \frac{1}{P^2_{\vec{a}_n}(z)}[E''_{\vec{a}_n}(z) + 4(1+z)^{-1}E'_{\vec{a}_n}(z) + 2(1+z)^{-2}E_{\vec{a}_n}(z)] \\ & - 2(1+z)^{-2}\frac{E_{\vec{a}_n}(z)}{P^2_{\vec{a}_n}(z)} - (1+z)^{-1}\frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}E_{\vec{a}_n}(z) - 2(1+z)^{-1}\frac{E'_{\vec{a}_n}(z)}{P^2_{\vec{a}_n}(z)} - \frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}E'_{\vec{a}_n}(z) \\ & - \frac{2}{(\kappa(z)-\lambda_n)\frac{1}{4}(1+z)^2P^2_{\vec{a}_n}(z)}\left[\frac{\mathrm{i}}{2}E'_{\vec{a}_n}(z) - \frac{\mathrm{i}}{4}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ = {}& \frac{1}{P^2_{\vec{a}_n}(z)}[E''_{\vec{a}_n}(z) + 2(1+z)^{-1}E'_{\vec{a}_n}(z)] - (1+z)^{-1}\frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}E_{\vec{a}_n}(z) - \frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}E'_{\vec{a}_n}(z) \\ & - \frac{2}{(\kappa(z)-\lambda_n)\frac{1}{4}(1+z)^2P^2_{\vec{a}_n}(z)}\left[\frac{\mathrm{i}}{2}E'_{\vec{a}_n}(z) - \frac{\mathrm{i}}{4}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ = {}& E_{\vec{a}_n}(z). \end{aligned}$$

Recalling the operator $\mathcal{L}_{u_n,\mathbb{D}}$ with $u_n = P_{\vec{a}_n}(z)$, noting that $\kappa(z) - \lambda_n = -\frac{2\mathrm{i}}{1+a_n}\frac{z-a_n}{1+z}$ for the κ-pair (λ_n, a_n) and

$$\begin{aligned} & -\frac{2}{(\kappa(z)-\lambda_n)\frac{1}{4}(1+z)^2P^2_{\vec{a}_n}(z)}\left[\frac{\mathrm{i}}{2}E'_{\vec{a}_n}(z) - \frac{\mathrm{i}}{4}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ & = \frac{2(1+a_n)}{(z-a_n)P^2_{\vec{a}_n}(z)}(1+z)^{-1}\left[E'_{\vec{a}_n}(z) - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ & = \frac{2}{(z-a_n)P^2_{\vec{a}_n}(z)}(1+a_n)\left(1 - \frac{z}{1+z}\right)\left[E'_{\vec{a}_n}(z) - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ & = \frac{2}{(z-a_n)P^2_{\vec{a}_n}(z)}\left(1 - \frac{z-a_n}{1+z}\right)\left[E'_{\vec{a}_n}(z) - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ & = \wp_{u_n,\mathbb{D}}(E_{\vec{a}_n}(z)) - \frac{2}{(1+z)P^2_{\vec{a}_n}(z)}\left[E'_{\vec{a}_n}(z) - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right], \end{aligned}$$

we obtain

$$\begin{aligned} -\frac{1}{C_1(1+z)}R(z) = {}& \mathcal{L}_{u_n,\mathbb{D}}(E_{\vec{a}_n}(z)) + 2(1+z)^{-1}\frac{E'_{\vec{a}_n}(z)}{P^2_{\vec{a}_n}(z)} - (1+z)^{-1}\frac{P'_{\vec{a}_n}(z)}{P^3_{\vec{a}_n}(z)}E_{\vec{a}_n}(z) \\ & + \wp_{u_n,\mathbb{D}}(E_{\vec{a}_n}(z)) - \frac{2}{(1+z)P^2_{\vec{a}_n}(z)}\left[E'_{\vec{a}_n}(z) - \frac{1}{2}\frac{P'_{\vec{a}_n}(z)}{P_{\vec{a}_n}(z)}E_{\vec{a}_n}(z)\right] \\ = {}& \mathcal{L}_{n,\mathbb{D}}(E_{\vec{a}_n}(z)) + \wp_{u_n,\mathbb{D}}(E_{\vec{a}_n}(z)) \\ = {}& \mathcal{L}_{n,\mathbb{D}}(E_{\vec{a}_n}(z)). \end{aligned}$$

This completes the proof of the theorem. □

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