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Two-dimensional adaptive Fourier decomposition

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One-dimensional adaptive Fourier decomposition, abbreviated as 1-D AFD, or AFD, is an adaptive representation of a physically realizable signal into a linear combination of parameterized Szegö and higher-order Szegö kernels of the context. In the present paper, we study multi-dimensional AFDs based on multivariate complex Hardy spaces theory. We proceed with two approaches of which one uses Product-TM Systems; and the other uses Product-Szegö Dictionaries. With the Product-TM Systems approach, we prove that at each selection of a pair of parameters, the maximal energy may be attained, and, accordingly, we prove the convergence. With the Product-Szegö dictionary approach, we show that pure greedy algorithm is applicable. We next introduce a new type of greedy algorithm, called Pre-orthogonal Greedy Algorithm (P-OGA). We prove its convergence and convergence rate estimation, allowing a weak-type version of P-OGA as well. The convergence rate estimation of the proposed P-OGA evidences its advantage over orthogonal greedy algorithm (OGA). In the last part, we analyze P-OGA in depth and introduce the concept P-OGA-Induced Complete Dictionary, abbreviated as Complete Dictionary . We show that with the Complete Dictionary P-OGA is applicable to the Hardy H^2 space on 2-torus. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: C ; ; ; ; F ; ; ; ; ; D ; C D

1. Preparation

F (1-D AFD) D **D** **C** 1,2 $L^2(\partial\mathbf{D})$

$$L^2(\partial\mathbf{D}) = H^2_+(\partial\mathbf{D}) \bigoplus H^2_-(\partial\mathbf{D}),$$

$H^2_+(\partial\mathbf{D})$ $H^2_-(\partial\mathbf{D})$ H^2- $H^2_\pm(\mathbf{D})$. $H^2_\pm(\mathbf{D})$ $H^2_\pm(\partial\mathbf{D})$ $H^2_\pm(\partial\mathbf{D})$ $L^2(\partial\mathbf{D})$) $(f \pm iHf)/2 \pm c_0/2, c_0 \in$ **C**, H 3. Ω 4 f Ω $\partial\Omega$. B $\tilde{f} = u + iv$, u v , v *the Hilbert transform of* u. u v $\partial\Omega$. 5 :

$$Hu(t) = \frac{1}{2\pi} {}_{\epsilon\to 0} \int_{|t-s|>\epsilon} \left(\frac{t-s}{2}\right) u(e^{is})ds$$

F :

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$$Hu(t) = \sum_{n=-\infty}^{\infty} (-i \quad (n))c_n e^{int}, \quad u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty,$$

(ξ) , $+1$ -1, , $\xi > 0$ $\xi < 0$; $(0) = 0$.
F , $L^2(\partial \mathbf{D})$, f Hf
. D $f^{\pm} = \frac{1}{2}(f \pm iHf) \pm c_0/2$. B $c_{-n} = \overline{c_n}$,

$$f = f^+ + f^-, \qquad f = 2 \quad f^+ - c_0. \tag{1.1}$$

$f \in L^2(\partial \mathbf{D})$,

$$_{r\to 1-} \frac{1}{2\pi i} \int_{\partial \mathbf{D}} \frac{f(\zeta)}{\zeta - re^{it}} d\zeta = \frac{1}{2}(f(e^{it}) + iHf(e^{it})) + \frac{c_0}{2},$$

, L

$$B_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z}\prod_{l=1}^{k-1}\frac{z-a_l}{1-\overline{a}_l z},\ k=1,2,\ldots,$$

$a_1, \ldots, a_k, \ldots$ B_k B $k-1$, $(k-1)$, $\mathcal{D}$, a a_k' , $\{B_k\}$ F , $\{z^{k-1}\}_{k=1}^{\infty}$, a_k' , $\{B_k\}$ H^p , $1 \leq p \leq \infty$, 11

$$\sum_{k=1}^{\infty}(1-|a_k|) = \infty \tag{1.4}$$

F $p=2$, $H^2(\mathbf{D})$ (1.4) $B_k(e^{it}) = \rho_k(t)e^{i(\psi_k(t)+\phi_{k-1}(t))}$, $e_{a_k}(e^{it}) = \rho_k(t)e^{i\psi_k(t)}, \rho_k(t) \geq 0, e^{i\phi_{k-1}(t)} = \prod_{l=1}^{k-1}\frac{e^{it}-a_l}{1-\overline{a}_l e^{it}}$, $\phi_{k-1}'(t) \geq 0$ $1+\psi_k'(t) > 0$ $t \in [0, 2\pi], k=1,2,\ldots, \phi_0' = 0$. , $a_1 = 0$, $\psi_k'(t) + \phi_{k-1}'(t) \geq 0$ t $k = 1,2,\ldots$ 1 . A B_k' , ( ) 1,6,7,13 15 . 1-D AFD. 2-D AFD 2 , 1-D AFD 1,2 . f $H^2(\mathbf{D})$. $f_1 = f$. F $a_1 \in \mathbf{D}$,

$$f(z) = \langle f_1, e_{a_1}\rangle e_{a_1}(z) + f_2(z)\frac{z-a_1}{1-\overline{a}_1 z}, \tag{1.5}$$

$$f_2(z) = \frac{f_1(z) - \langle f_1, e_{a_1}\rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a}_1 z}}.$$

B e_{a_1} $H^2(\mathbf{D})$,

$$\langle f, e_{a_1}\rangle = \sqrt{1-|a_1|^2}f(a_1), \qquad f_1(a_1) - \langle f_1, e_{a_1}\rangle e_{a_1}(a_1) = 0.$$

$f_2 \in H^2(\mathbf{D})$ f_1 f_2 *generalized backward shift via* a_1; f_2, *reduced remainder*, *the generalized backward shift transform* f_1 a_1.

$$S(f)(z) = \sum_{k=0}^{\infty}c_{k+1}z^k = \frac{f(z)-f(0)}{z},$$

$f(z) = \sum_{k=0}^{\infty}c_k z^k$. $f(0) = \langle f, e_0\rangle e_0(z)$, S 0.

B (1.5) ,

$$\|f\|^2 = \|\langle f_1, e_{a_1}\rangle e_{a_1}\|^2 + \|f_2\frac{(\cdot)-a_1}{1-\overline{a}_1(\cdot)}\|^2 = |\langle f_1, e_{a_1}\rangle|^2 + \|f_2\|^2.$$

$a_1 \in \mathbf{D}$ $\langle f_1, e_{a_1}\rangle e_{a_1}(z)$ $\|f\|^2$. B e_{a_1}

$$|\langle f_1, e_{a_1}\rangle|^2 = (1-|a_1|^2)|f_1(a_1)|^2, \tag{1.6}$$

, $a_1 \in \mathbf{D}$

$$a_1 = \{(1-|a|^2)|f_1(a)|^2 \ :\ a \in \mathbf{D}\}.$$

(1 , 2.3 3.1) *Maximal Selection Principle*. a_1, f_2, . A n ,

$$f(z) = \sum_{k=1}^{n} \langle f_k, e_{a_k} \rangle B_k(z) + f_{n+1} \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z},$$

$k = 1, \ldots, n,$

$$a_k = \quad \{(1 - |a|^2)|f_k(a)|^2 \; : \; a \in \mathbf{D}\}, \tag{1.7}$$

$k = 2, \ldots, n+1,$

$$f_k(z) = \frac{f_{k-1}(z) - \langle f_{k-1}, e_{a_{k-1}} \rangle e_{a_{k-1}}(z)}{\frac{z - a_{k-1}}{1 - \bar{a}_{k-1} z}}.$$

B

$$\|f - \sum_{k=1}^{n} \langle f_k, e_{a_k} \rangle B_k(z)\|^2 = \|f\|^2 - \sum_{k=1}^{n} |\langle f_k, e_{a_k} \rangle|^2 = \|f_{k+1}\|^2.$$

$$\lim_{n \to \infty} \|f_{k+1}\| = 0$$

1 ,

$$f(z) = \sum_{k=1}^{\infty} \langle f_k, e_{a_k} \rangle B_k(z). \tag{1.8}$$

f (1.8) F (AFD) f.

:

$$\langle f_k, e_{a_k} \rangle = \langle g_k, B_k \rangle = \langle f, B_k \rangle, \tag{1.9}$$

g_k *orthogonal standard remainder*

$$f = \sum_{i=1}^{k-1} \langle f, B_i \rangle B_i(z) + g_k(z). \tag{1.10}$$

$$g_k(z) = f_k(z) \prod_{l=1}^{k-1} \frac{z - a_l}{1 - \bar{a}_l z}, \qquad f_k = S_{a_{k-1}} \cdots S_{a_1} f(z). \tag{1.11}$$

2 :

$$H^2(\mathcal{D}, M) := \{f \in H^2(D) : f = \sum_{k=1}^{\infty} c_k e_k, e_k \in \mathcal{D}, \sum_{k=1}^{\infty} |c_k| \leq M\}, \quad 0 < M < \infty. \tag{1.12}$$

2 :

Theorem 1.1

$\mathcal{D}$ $H^2(D)$. $f \in H^2(\mathcal{D}, M)$, AFD,

$$\| g_k \| \leq \frac{M}{\sqrt{k}}.$$

$H^2(\mathcal{D}, M)$ AFD .

Remark 1

1-D AFD (C AFD AFD) AFD 16 , AFD 17 , AFD 18 . E AFD , AFD . C AFD n 17 .

Remark 2

F 19 22 .

Remark 3

A $a_1 = 0$ AFD

Remark 4

F *Pre-orthogonal Greedy Algorithm (P-OGA)* A AFD A A A). AFD, n, a_n f_n A g_n. A g_n

$$1, \frac{1}{1-\overline{a}_1 z}, \dots, \frac{1}{1-\overline{a}_n z}, \quad n = 1, 2, \dots,$$

a_k' 1-D AFD, (1.3),

$$1, \dots, z^{m_0-1}, \frac{1}{1-\overline{a}_1 z}, \dots, \frac{1}{(1-\overline{a}_1 z)^{m_1}}, \dots, \frac{1}{1-\overline{a}_n z}, \dots, \frac{1}{(1-\overline{a}_n z)^{m_n}}, \quad n = 1, 2, \dots, \tag{1.13}$$

a_n m_n (2 3). AFD

Remark 5

$a_1, \dots, a_n, \dots$ (1.4). B $\phi(z)$ $a_1, \dots, a_k, \dots$, 5

$$H^2 = \overline{\{B_k\}} \oplus \phi H^2, \tag{1.14}$$

$\overline{\{B_k\}}$ *backward shift invariant subspace* ϕH^2 *shift invariant subspace* H^2 , f B B 21 $1 < p < \infty$, (1.4) $\{B_k\}$ $L^p(\partial \mathbf{D})$ $\{B_k\}$ 22 AFD AFD F E $\mathbf{R}^n$, C $\mathbf{R}^n \subset \mathbf{C}^n$. $f(x_1, \dots, x_n)$, $f(z_1, \dots, z_n)$. f f f $z_k, k = 1, 2, \dots, n$. C $f(x_1, \dots, x_n)$ $f(x_0, x_1, \dots, x_n)$. 23, 24 C 23, 24 C 25, 26 C 1-D AFD C 27 AFD 28 C B $\mathbf{R}^n$ AFD 29 -C $\mathbf{R}^n$, C F

F 1-D AFD B D 2 3. 1-D D A A A. $\rho < 1$ (3.24). (3.3) A A). 4, A *Induced Complete Dictionary*, C D A. A 2- C D D . F 1-D C D D A AFD.

2. 2-D AFD of the Product-TM System type

. F **a** $\{a_n\}$ $a_1, a_2, \ldots$ **D**, $\mathcal{B}^{\mathbf{a}}$ **a**,

$$\mathcal{B}^{\mathbf{a}} = \{B_{\{a_1,\ldots,a_n\}}\} = \{B_n^{\mathbf{a}}\},$$

$$B_n^{\mathbf{a}}(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z}\prod_{l=1}^{n-1}\frac{z-a_l}{1-\bar{a}_l z}, \quad n = 1, 2, \ldots$$

a , $\mathbf{a} = \{a_1, \ldots, a_N\}$, $\mathcal{B}^{\mathbf{a}}$ $\mathcal{B}_N^{\mathbf{a}}$.
T $\partial\mathbf{D}$, $L^2(\mathbf{T}^2)$ 2- ,

$$\langle f, g\rangle = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(e^{it}, e^{is})\bar{g}(e^{it}, e^{is})dtds.$$

F , $f \in L^2(\mathbf{T}^2)$

$$f(e^{it}, e^{is}) = \sum_{-\infty<k,l<\infty} c_{kl}e^{i(kt+ls)} \qquad L^2 - ,$$

$$\sum_{-\infty<k,l<\infty} |c_{kl}|^2 < \infty, \quad c_{kl} = \langle f, e_{kl}\rangle, \quad e_{kl}(t,s) = e^{ikt}e^{ils}.$$

D

$$H^2(\mathbf{T}^2) = \{f \in L^2(\mathbf{T}^2) : f(e^{it}, e^{is}) = \sum_{k,l\geq 0} c_{kl}e^{i(kt+ls)}\}.$$

, $L^2(\mathbf{T}^2)$ $H^2(\mathbf{T}^2)$, , H^2 L^2. H^2 L^2. D $H^2(\mathbf{D}^2)$, $\mathbf{D}\times\mathbf{D}$

$$\sup_{0<r,s<1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(re^{it}, se^{iu})|^2 dtdu < \infty.$$

$f \in H^2(\mathbf{D}^2)$,

$$\lim_{z\to e^{it}; w\to e^{is}} f(z,w) \qquad (e^{it}, e^{is}) \in \mathbf{T}^2,$$

$z \to e^{it}$ $w \to e^{is}$, $\mathbf{T}^2$ $H^2(\mathbf{T}^2)$. $f \in H^2(\mathbf{D}^2)$ $H^2(\mathbf{T}^2)$, , , . F , H^2 $H^2(\mathbf{D}^2)$ $H^2(\mathbf{T}^2)$. $\mathbf{T}^2$, *characteristic boundary* $\mathbf{D}^2$, . D , , D(D) 7508146.279(0 D10069(0 D10069(70.02140186.97179.956 (2) 8.95.

F , $f \in L^2$. D

$$f^{+,+}\left(e^{it},e^{is}\right) = \sum_{k,l\geq 0} c_{lk} e^{i(kt+ls)},$$

$$f^{+,-}\left(e^{it},e^{is}\right) = \sum_{k,-l\geq 0} c_{lk} e^{i(kt+ls)},$$

$$f^{-,+}\left(e^{it},e^{is}\right) = \sum_{-k,l\geq 0} c_{lk} e^{i(kt+ls)},$$

$$f^{-,-}\left(e^{it},e^{is}\right) = \sum_{-k,-l\geq 0} c_{lk} e^{i(kt+ls)}.$$

A (1.1),

Theorem 2.1

$f \in L^2$

$$f\left(e^{it},e^{is}\right) = 2\left\{f^{+,+}\right\}\left(e^{it},e^{is}\right) + 2\left[f\left(e^{i(\cdot)},e^{-i(\cdot)}\right)\right]^{+,+}\left(e^{it},e^{-is}\right) - 2\left\{F^{+}\right\}\left(e^{it}\right) - 2\left\{G^{+}\right\}\left(e^{is}\right) + c_{00}.$$

Proof

$$f\left(e^{it},e^{is}\right) + F\left(e^{it}\right) + G\left(e^{is}\right) + c_{00} = f^{+,+}\left(e^{it},e^{is}\right) + f^{+,-}\left(e^{it},e^{is}\right) + +f^{-,+}\left(e^{it},e^{is}\right) + f^{-,-}\left(e^{it},e^{is}\right),$$

$$F\left(e^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it},e^{is}\right) ds, \qquad G\left(e^{is}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it},e^{is}\right) dt.$$

$$f\left(e^{it},e^{is}\right) = f^{+,+}\left(e^{it},e^{is}\right) + f^{+,-}\left(e^{it},e^{is}\right) + f^{-,+}\left(e^{it},e^{is}\right) + +f^{-,-}\left(e^{it},e^{is}\right) - F\left(e^{it}\right) - G\left(e^{is}\right) - c_{00}.$$

$$\left[f\left(e^{i(\pm\cdot)},e^{i(\pm\cdot)}\right)\right]^{+,+}\left(e^{i(\pm t)},e^{i(\pm s)}\right) = f^{\pm,\pm}\left(e^{it},e^{is}\right), \quad \left[f\left(e^{i(\pm\cdot)},e^{i(\mp\cdot)}\right)\right]^{+,+}\left(e^{i(\pm t)},e^{i(\mp s)}\right) = f^{\pm,\mp}\left(e^{it},e^{is}\right).$$

B f , $f^{+,+}$ $f^{-,-}$, $f^{+,-}$ $f^{-,+}$

$$f^{+,+} + f^{-,-} = 2\left\{f^{+,+}\right\}$$

$$f^{+,-} + f^{-,+} = 2\left\{f^{+,-}\right\}.$$

$$\begin{aligned}
f\left(e^{it},e^{is}\right) &= 2\left\{f^{+,+}\right\}\left(e^{it},e^{is}\right) + 2\left\{f^{+,-}\right\}\left(e^{it},e^{is}\right) - F\left(e^{it}\right) - G\left(e^{is}\right) - c_{00}\\
&= 2\left\{f^{+,+}\right\}\left(e^{it},e^{is}\right) + 2\left[f\left(e^{i(\cdot)},e^{-i(\cdot)}\right)\right]^{+,+}\left(e^{it},e^{-is}\right) - F\left(e^{it}\right) - G\left(e^{is}\right) - c_{00}\\
&= 2\left\{f^{+,+}\right\}\left(e^{it},e^{is}\right) + 2\left[f\left(e^{i(\cdot)},e^{-i(\cdot)}\right)\right]^{+,+}\left(e^{it},e^{-is}\right) - 2\left\{F^{+}\right\}\left(e^{it}\right) - 2\left\{G^{+}\right\}\left(e^{is}\right) + c_{00}.
\end{aligned}$$

□

$f^{+,+}(e^{it},e^{is}), f^{+,-}(e^{it},e^{-is}), f^{-,+}(e^{-it},e^{is}), f^{-,-}(e^{-it},e^{-is})$ H^2. $f \in L^2$ 1-D

Theorem 2.2

$\mathcal{B}_N^{\mathbf{a}}$ $\mathcal{B}_M^{\mathbf{b}}$ $\mathcal{B}_N^{\mathbf{a}} \otimes \mathcal{B}_M^{\mathbf{b}}$ $L^2(\mathbf{T}^2)$. $\mathcal{B}^{\mathbf{a}}$ $\mathcal{B}^{\mathbf{b}}$ $H^2(\mathbf{T})$, $\mathcal{B}^{\mathbf{a}} \otimes \mathcal{B}^{\mathbf{b}}$ $H^2(\mathbf{T}^2)$.

Proof

$\sum_{k=1}^{K} f_k(z)g_k(w)$ $H^2(\mathbf{T}^2)$. $H^2(\mathbf{T}^2)$. $\mathcal{B}^{\mathbf{a}}$ $\mathcal{B}^{\mathbf{b}}$ $H^2(\mathbf{T})$, $\mathcal{B}^{\mathbf{a}} \otimes \mathcal{B}^{\mathbf{b}}$ $\sum_{k=1}^{K} f_k(z)g_k(w)$ $H^2(\mathbf{T}^2)$.

D $f \in H^2(\mathbf{T}^2)$,

$$S_n(f) = \sum_{1\le k,l\le n} \langle f, B_k^{\mathbf{a}} \otimes B_l^{\mathbf{b}} \rangle B_k^{\mathbf{a}} \otimes B_l^{\mathbf{b}} = \sum_{k=1}^{n} D_n(f),\ D_n(f) = S_n(f) - S_{n-1}(f),\ S_0(f) = 0, \tag{2.15}$$

$$n = 1, 2, \ldots$$

□

$D_n(f)$ *n-partial sum difference* $2n-1$

Theorem 2.3 ( )

F $f \in H^2$ $a_1, \ldots, a_{n-1}$ $b_1, \ldots, b_{n-1}$ **D** a_n, b_n **D**

$$\|D_n(f)\|^2 = \sum_{\{k,l\}=n} |\langle f, B_k^{\mathbf{a}} \otimes B_l^{\mathbf{b}} \rangle|^2 \tag{2.16}$$

a_n, b_n

Proof

$f \in H^2$: ( ) $|a_n| \to 1$ $|b_n| \to 1$, $a_1, \ldots, a_{n-1}$ $b_1, \ldots, b_{n-1}$ $a_1, \ldots, a_{n-1}$ $b_1, \ldots, b_{n-1}$,

$$\lim_{|a_n|\to 1, |b_n|\to 1} \|D_n(f)\|^2 = 0;$$

( ) $|a_n|$ $|b_n|$ 1, (2.16) (a_n, b_n) ; □

( ). A C $2n-1$ D_n, $\epsilon > 0$, P

$$\|D_n(f-P)\|^2 \le \epsilon$$

$a_1, \ldots, a_{n-1}, a_n$ $b_1, \ldots, b_{n-1}, b_n$. P,

$$\lim_{|a_n|\to 1, |b_n|\to 1} \|D_n(P)\|^2 = 0.$$

D_n (2.15), $|a_n| \to 1$ $|b_n| \to 1$,

$$|\langle P, B_n^{\mathbf{a}} \otimes B_l^{\mathbf{b}} \rangle|^2 \to 0, \qquad 1 \le l \le n \tag{2.17}$$

$$|\langle P, B_k^{\mathbf{a}} \otimes B_n^{\mathbf{b}} \rangle|^2 \to 0, \qquad 1 \le k < n. \tag{2.18}$$

D $S_a^{(1)}$ a z, $S_b^{(2)}$ w.
B (1.11), $e_{a_n} \otimes e_{b_l}$,

$$\begin{aligned} \langle P, B_n^{\mathbf{a}} \otimes B_l^{\mathbf{b}} \rangle &= \left\langle \prod_{k=1}^{n-1} S_{a_k}^{(1)} \prod_{k=1}^{l-1} S_{b_k}^{(2)}(P), e_{a_n} \otimes e_{b_l} \right\rangle \\ &= \sqrt{1-|a_n|^2}\sqrt{1-|b_l|^2} \prod_{k=1}^{n-1} S_{a_k}^{(1)} \prod_{k=1}^{l-1} S_{b_k}^{(2)}(P)(a_n, b_l) \\ &\to 0, \qquad |a_n| \to 1, \end{aligned} \tag{2.19}$$

(2.18).

$$\lim_{|a_n|\to 1, |b_n|\to 1} \|D_n(P)\|^2 = 0.$$

( ) $|a_n| \to 1$. B (2.19), (2.17) $2n-1$
$D_n(f)$ $|a_n| \to$

B

$$\sum_{k=1}^{n} \|D_k(f)\|^2 \leq \|f\|^2.$$

A

$$_{n\to\infty} \sum_{k=n+1}^{\infty} \|D_k(f)\|^2 = 0.$$

(a_n, b_n) 2.3.

Theorem 2.4

$f \in H^2(\mathbf{T}^2)$. F k_0 $a_1, b_1, \ldots, a_{k_0-1}, b_{k_0-1}$, $(a_{k_0}, b_{k_0}), (a_{k_0+1}, b_{k_0+1}), \ldots,$

$$_{n\to\infty} \|f - S_n(f)\|^2 = 0.$$

L^2-

$$f = {}_{n\to\infty} S_n(f).$$

Proof

. A

$$f = \sum_{k=1}^{\infty} D_k(f) + h, \quad h \neq 0,$$

h H^2 $D_k(f)$.

$$\|h\|^2 = \|f\|^2 - \sum_{k=1}^{\infty} \|D_k(f)\|^2 > 0.$$

B C $\tilde{a}, \tilde{b}$ **D**,

$$\langle h, e_{\{\tilde{a}\}} \otimes e_{\{\tilde{b}\}} \rangle = \sqrt{1 - |\tilde{a}|^2}\sqrt{1 - |\tilde{b}|^2}\, h(\tilde{a}, \tilde{b})$$

$\tilde{a}, \tilde{b}$ $\langle h, e_{\{\tilde{a}\}} \otimes e_{\{\tilde{b}\}} \rangle \neq 0$. D

$$\tilde{X} = \overline{} \{\tilde{\mathcal{B}}^{\mathbf{a}} \otimes \tilde{\mathcal{B}}^{\mathbf{b}}\},$$

$\tilde{\mathcal{B}}^{\mathbf{a}}$ $\{\tilde{a}, a_1, \ldots, a_n, \ldots\}$; $\tilde{\mathcal{B}}^{\mathbf{b}}$.
n- $\tilde{D}_n$, $\tilde{a}, \tilde{b}, a_1, b_1, \ldots, a_{n-1}, b_{n-1}$.

□

D $h/\tilde{X}$ h $\tilde{X}$.

$$\|h/\tilde{X}\|_2 = \delta > 0.$$

$$\|h/\tilde{X}\|_2^2 \geq \sum_{k=1}^{\infty} \|\tilde{D}_k\|_2^2 \geq \|\tilde{D}_1\|_2^2 = |\langle h, e_{\tilde{a}} \otimes e_{\tilde{b}} \rangle|^2 > 0.$$

M,

$$\tilde{X}_M = \quad \{\tilde{\mathcal{B}}_M^{\mathbf{a}} \otimes \tilde{\mathcal{B}}_M^{\mathbf{b}}\} \qquad X_M = \quad \{\mathcal{B}_M^{\mathbf{a}} \otimes \mathcal{B}_M^{\mathbf{b}}\},$$

$\tilde{\mathcal{B}}_M^{\mathbf{a}}$ $\tilde{\mathcal{B}}_M^{\mathbf{b}}$ $\{\tilde{a}, a_1, \ldots, a_{M-1}\}$ $\{\tilde{b}, b_1, \ldots, b_{M-1}\}$.
B

$$\|h/\tilde{X} - h/\tilde{X}_M\|^2 = \sum_{k=M+2}^{\infty} \|\tilde{D}_k\|^2 \to 0,$$

$$_{M\to\infty} h/\tilde{X}_M = h/\tilde{X}.$$

M

$$\|h/\tilde{X}_M\|^2 > \delta/2 \qquad \left\| \sum_{k=M}^{\infty} D_k(f) \right\|^2 < \delta/8.$$

F M,

Proof

$\tilde{g}$

$$_{|a|\to 1-} \quad _{|b|\to 1-} |\langle \tilde{g}, e_a \otimes e_b\rangle|^2 = 0.$$

□

B

$$_{|a|\to 1-} \quad _{|b|\to 1-} \|\tilde{g} - \langle \tilde{g}, e_a \otimes e_b\rangle e_a \otimes e_b\| = \|\tilde{g}\|. \tag{3.21}$$

$P_r \otimes P_s$ 24 , $r, s \in [0, 1)$. F $\epsilon > 0$, r s 1 L^2-

$$\begin{aligned} \|\tilde{g}\| &\geq \|\tilde{g} - \langle \tilde{g}, e_a \otimes e_b\rangle e_a \otimes e_b\| \\ &\geq \|(P_r \otimes P_s) * [\tilde{g} - \langle \tilde{g}, e_a \otimes e_b\rangle e_a \otimes e_b]\| \\ &\geq \|(P_r \otimes P_s) * \tilde{g}\| - |\langle \tilde{g}, e_a \otimes e_b\rangle| \|(P_r \otimes P_s) * (e_a \otimes e_b)\| \\ &\geq (1-\epsilon)\|\tilde{g}\| - \|\tilde{g}\| \|(P_r \otimes P_s) * (e_a \otimes e_b)\|. \end{aligned} \tag{3.22}$$

r s, $e_a \otimes e_b \in H^2$, $z = re^{it}, w = se^{iu}$,

$$(P_r \otimes P_s) * (e_a \otimes e_b)(e^{it}, e^{iu}) = e_a(z)e_b(w).$$

$$\begin{aligned} \|(P_r \otimes P_s) * (e_a \otimes e_b)\|^2 &= \frac{1}{(2\pi)^2}\int_0^{2\pi} \frac{1-|a|^2}{|1-\bar{a}re^{it}|^2}dt \int_0^{2\pi} \frac{1-|b|^2}{|1-\bar{b}se^{iu}|^2}du \\ &= \frac{1-|a|^2}{1-r^2|a|^2}\,\frac{1-|b|^2}{1-s^2|b|^2}. \end{aligned}$$

$|a| \to 1$ $|b| \to 1$, (3.22)

$$\|\tilde{g}\| \geq \|\tilde{g} - \langle \tilde{g}, e_a \otimes e_b\rangle e_a \otimes e_b\| \geq (1-2\epsilon)\|\tilde{g}\|.$$

(3.21)

B D A 30, 32 34 ,

$$f = \sum_{k=1}^{\infty} \langle g_k, e_{a_k} \otimes e_{b_k}\rangle e_{a_k} \otimes e_{b_k}.$$

A A 30, 33, 34 . A, A. A, A.

$\mathcal{A}$ $a \in \mathcal{A}$ $\|e_a\| = 1$, $\overline{\mathcal{A}} =$

$$f = \sum_{k=1}^{n-1} \langle f, B_k\rangle B_k + g_n, \tag{3.23}$$

$\{B_1, \ldots, B_k\}$ $\{a_1, \ldots, a_k\}$, k, a_k *pre-orthogonal ρ-Maximal Selection Principle*

$$|\langle g_k, B_k\rangle| \geq \rho \cdot \{|\langle g_k, B_k^a\rangle| : a \in \mathcal{A}\}, \quad \rho \in (0, 1], \tag{3.24}$$

$\{B_1, \ldots, B_{k-1}, B_k^a\}$ $\{a_1, \ldots, a_{k-1}, a\}$. *Weak Pre-orthogonal Greedy Algorithm,* A. $\rho = 1$, *Pre-orthogonal Greedy Algorithm,* *P-OGA.* *Weak Orthogonal Greedy Algorithm* (WOGA, 30), a_k

$$|\langle g_k, a_k\rangle| \geq \rho \cdot \{|\langle g_k, a\rangle| : a \in \mathcal{A}\}, \quad \rho \in (0, 1]. \tag{3.25}$$

A (A) A(A) A A A $a_k \in \mathcal{A}$, $|\langle g_k, a_k\rangle|$ $\{|\langle g_k, a\rangle| : a \in \mathcal{A}\}$,

$$\{|\langle g_k, B_k^{a}\rangle| : a \in \mathcal{A}\} \geq |\langle g_k, B_k^{a_k}\rangle|.$$

A 3.3.

A A,

$$\langle f, B_k\rangle = \langle g_k, B_k\rangle.$$

$$\left\| f - \sum_{k=1}^{n} \langle f, B_k\rangle B_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} |\langle f, B_k\rangle|^2,$$

B

$$\sum_{k=1}^{\infty} |\langle f, B_k\rangle|^2 \leq \|f\|^2.$$

Theorem 3.2

$\mathcal{A}$. F $f \in$, $a_1, \dots, a_n, \dots$ $\mathcal{A}$ ρ-

$$f = \sum_{k=1}^{\infty} \langle f, B_k\rangle B_k,$$

k, $\{B_1, \dots, B_k\}$ $\{a_1, \dots, a_k\}$.

ρ- .

Proof

. A $f = \sum_{k=1}^{\infty} \langle f, B_k\rangle B_k + h$, $h \neq 0$, $h \perp \overline{\mathcal{B}}$, $\mathcal{B} = \{a_1, \dots, a_n, \dots\}$. B $\overline{\mathcal{A}} =$, $b \in \mathcal{A}$ $\langle h, e_b\rangle \neq 0$. D $\mathcal{B}^b = \{b, a_1, \dots, a_n, \dots\}$. , $h/\overline{\mathcal{B}^b} \neq 0$. $\|h/\overline{\mathcal{B}^b}\| = \delta\ (> 0)$. D $\mathcal{B}_n = \{a_1, \dots, a_n\}$, $\mathcal{B}^b_{n+1} = \{b, a_1, \dots, a_n\}$.

$$_{n \to \infty} h/ \mathcal{B}^b_{n+1} = h/\overline{\mathcal{B}^b}.$$

F N

$$\|h/ \mathcal{B}^b_{N+1}\| > \delta/2, \qquad \left\| \sum_{k=N+1}^{\infty} \langle f, B_k\rangle B_k \right\| < \delta/2^m,$$

m

$$\begin{aligned} f &= \sum_{k=1}^{N} \cdot + \sum_{k=N+1}^{\infty} \cdot + h \\ &= \sum_{k=1}^{N} \cdot + g_{N+1}. \end{aligned}$$

□

$$\begin{aligned} |\langle f, B_{N+1}\rangle| &= |\langle g_{N+1}, B_{N+1}\rangle| \\ &= |\langle \sum_{k=N+1}^{\infty} \cdot, B_{N+1}\rangle| \\ &\leq \| \sum_{k=N+1}^{\infty} \cdot \| \\ &\leq \delta/2^m. \end{aligned}$$

$\{B_1, \ldots, B_N, B^b_{N+1}\}$ $\{a_1, \ldots, a_N, b\}$,

$$
\begin{aligned}
|\langle f, B^b_{N+1}\rangle| &= |\langle g_{N+1}, B^b_{N+1}\rangle| \\
&= |\langle h + \sum_{N+1}^{\infty} \cdot, B^b_{N+1}\rangle| \\
&\geq |\langle h, B^b_{N+1}\rangle| - |\langle \sum_{N+1}^{\infty} \cdot, B^b_{N+1}\rangle| \\
&= \|h / \quad \mathcal{B}^b_{n+1}\| - |\langle \sum_{N+1}^{\infty} \cdot, B^b_{N+1}\rangle| \\
&\geq \delta/2 - \delta/2^m \\
&= \frac{(2^{m-1} - 1)\delta}{2^m}.
\end{aligned}
$$

$$
|\langle f, B_{N+1}\rangle| / \quad \{|\langle f, B^a_{N+1}\rangle| : a \in \mathcal{A}\} < \frac{1}{2^{m-1} - 1}.
$$

m $\frac{1}{2^{m-1}-1} < \rho$, . A $\{B_1, \ldots, B_{n-1}\}$ $\{a_1, \ldots, a_{n-1}\}$. a_n $B_1, \ldots, B_{n-1}$, a_n $\sum_{k=1}^{n-1} \langle a_n, B_k\rangle B_k$, n $a_n - \sum_{k=1}^{n-1} \langle a_n, B_k\rangle B_k$ (3.23). a_n $\{a_1, \ldots, a_{n-1}\}$. $Q_{\{a_1, \ldots, a_{n-1}\}}$.

$$
Q_{\{a_1, \ldots, a_{n-1}\}}(a_n) = a_n - \sum_{k=1}^{n-1} \langle a_n, B_k\rangle B_k, \quad B_n = \frac{Q_{\{a_1, \ldots, a_{n-1}\}}(a_n)}{\|Q_{\{a_1, \ldots, a_{n-1}\}}(a_n)\|}.
$$

$Q_{\{a_1, \ldots, e_{n-1}\}}$ Q_{n-1}.

$$
g_n = Q_{\{a_1, \ldots, a_{n-1}\}}(f), \tag{3.26}
$$

g_n n f $\{B_1, \ldots, B_{n-1}\}$ (3.23). f g $\langle f, Q_{\{a_1, \ldots, a_{n-1}\}}(g)\rangle = \langle Q_{\{a_1, \ldots, e_{n-1}\}}(f), g\rangle$ $\langle f, g\rangle - \sum_{k=1}^{n-1} \langle f, B_k\rangle \langle B_k, g\rangle$.
B $\|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\| \neq 0$,

$$
\begin{aligned}
|\langle g_n, B^a_n\rangle| &= \frac{1}{\|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\|} |\langle Q_{\{a_1, \ldots, a_{n-1}\}}(f), Q_{a_1, \ldots, a_{n-1}}(a)\rangle| \\
&= \frac{1}{\|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\|} |\langle Q^2_{\{a_1, \ldots, a_{n-1}\}}(f), a\rangle| \\
&= \frac{1}{\|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\|} |\langle Q_{\{a_1, \ldots, a_{n-1}\}}(f), a\rangle| \\
&= \frac{1}{\|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\|} |\langle g_n, a\rangle|.
\end{aligned} \tag{3.27}
$$

$a_1, \ldots, a_{n-1}$ $\mathcal{A}$, $a \in \mathcal{A}$,

$$
r_n(a) = \|Q_{\{a_1, \ldots, a_{n-1}\}}(a)\|. \tag{3.28}
$$

$r_n(a) \leq 1$. $r_n(a) = 1$ a $\{a_1, \ldots, a_{n-1}\}$. $r_n(a) = 0$ $a \in$ $\{a_1, \ldots, a_{n-1}\}$. ρ

$$
|\langle g_n, B^a_n\rangle| = \frac{1}{r_n(a)} |\langle g_n, a\rangle| \geq |\langle g_n, a\rangle|.
$$

33 (2),

$$H^2(\mathcal{A}, M) = \{f \in H^2 \,:\, f = \sum_{k=1}^{\infty} c_k a_k, \sum_{k=1}^{\infty} |c_k| \leq M\}.$$

:

Theorem 3.3

$f \in H^2(\mathcal{A}, M)$. D g_m f $\{B_1, \ldots, B_{m-1}\}$ $\{a_1, \ldots, a_{m-1}\}$ ρ $R_m =$ $\{r_1, \ldots, r_m\}$, $r_n =$ $_k\{r_n(b_k)\}$, $r_n(b_k)$, $a_1, \ldots, a_{n-1}$ b_k, (3.28),

$$\|g_m\| \leq \frac{R_m M}{\rho} \frac{1}{\sqrt{m}}.$$

33

Lemma 3.4

$\{d_n\}_{n=1}^{m}$ m

$$d_1 \leq A_m, \qquad d_{n+1} \leq d_n \left(1 - \frac{d_n}{A_m}\right).$$

$$d_m \leq \frac{A_m}{m}.$$

m $n \leq m$, $A_m \leq A$, m,

$$d_m \leq \frac{A}{m}.$$

Proof of Theorem 3.3

A $f = \sum_k c_k b_k$ $\sum_k |c_k| \leq M$.

$$\|g_{m+1}\|^2 = \|g_m\|^2 - |\langle g_m, B_m \rangle|^2.$$

: F $n \leq m$,

$$\begin{aligned}
|\langle g_n, B_n \rangle| &\geq \rho \cdot_{a \in \mathcal{A}} |\langle g_n, B_n^a \rangle| \\
&\geq \rho \cdot_k |\langle g_n, B_n^{b_k} \rangle| \\
&= \rho \cdot_k \frac{|\langle g_n, b_k \rangle|}{r_n(b_k)} \\
&\geq \frac{\rho}{r_n} \cdot_k |\langle g_n, b_k \rangle| \\
&\geq \frac{\rho}{r_n M} |\langle g_n, \sum_k c_k b_k \rangle| \\
&= \frac{\rho}{r_n M} |\langle g_n, f \rangle| \\
&\geq \frac{\rho}{R_m M} \|g_n\|^2
\end{aligned}$$

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$$\underset{|a|\to 1}{}\ \underset{|b|\to 1}{}\ \|g-\langle g, N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b\rangle N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b\|^2=0. \tag{4.38}$$

(3.22),

$$\underset{|a|\to 1}{}\ \underset{|b|\to 1}{}\ \|(P_r\otimes P_s)*(N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b)\|\to 0. \tag{4.39}$$

, $0<r<1$ $0<s<1$, $N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b\in H^2$, , $z=re^{it}, w=se^{iu}$,

$$(P_r\otimes P_s)*(N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b)(e^{it},e^{iu})=(N(a,k)\partial_{\overline{a}}^{k}e_a)(z)(N(b,l)\partial_{\overline{b}}^{l}e_b)(w).$$

,

$$\left\|(P_r\otimes P_s)*(N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b)\right\|^2=\frac{N(a,k)}{N(ra,k)}\frac{N(b,l)}{N(sb,l)}\to 0, \qquad |a|\to 1 \qquad |b|\to 1.$$

(3.26),

$$1=\left\|N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b\right\|^2=\sum_{k=1}^{n-1}\left|\left\langle N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b, B_k\right\rangle\right|^2+\left\|Q_{n-1}\left(N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b\right)\right\|^2.$$

B (4.36),

$$\underset{|a|\to 1}{}\ \underset{|b|\to 1}{}\ \sum_{k=1}^{n-1}\left|\left\langle N(a,k)\partial_{\overline{a}}^{k}e_a\otimes N(b,l)\partial_{\overline{b}}^{l}e_b, B_k\right\rangle\right|^2=0.$$

(4.37). , (4.35). , $\widetilde{\mathcal{D}^2}$, $\mathcal{D}^2$, $|\langle g_n, B_n^a\rangle|, a\in\widetilde{\mathcal{A}}$

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