A CLASS OF UNBOUNDED FOURIER MULTIPLIERS ON THE UNIT COMPLEX BALL

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Abstract. In this paper, we introduce a class of Fourier multiplier operators M on $-$complex unit sphere, where the symbol $\in H(S_\omega)$. We obtained the Sobolev boundedness of M. Our result implies that the operators M take a role of fractional differential operators on $\partial\mathbb{B}$.

1. Introduction

In this paper, we introduce a class of unbounded holomorphic Fourier multipliers M on $-$complex unit sphere. We further study the boundedness of M on Sobolev spaces. Our results generalize the theory of Fourier multipliers on Lipschitz curves in $\mathbb{C}$ to $-$complex unit sphere $\mathbb{B}$. We refer the reader to Gaudry-Qian-Wang [3], McIntosh-Qian [8], and Qian [9, 10] for further information on multipliers on Lipschitz curves.

Our motivation originates from the following example on the unit sphere in $\mathbb{C}$. The explicit formula of the Cauchy-Szegö kernel

$$H(,\bar{\xi}) = \frac{1}{\omega_{2\ -1}} \frac{1}{(1- \ \overline{\xi'})}.$$

Let $\{\ \}$ denote the orthonormal system in the space of holomorphic functions in $\mathbb{B}$. The following result is well-known.

$$H(,\bar{\xi}) = \sum_{=0}^{\infty} \sum_{=1}^{N} (\ ) \overline{\ (\xi)}, \quad \in \mathbb{B} , \xi \in \partial\mathbb{B} \tag{1.1}$$

See Theorem 2.1 and (2.4) below for details. Formally, (1.1) can be seen as the special case of (1.2) below. Let S_ω be the sector defined as

$$S_\omega = \Big\{ \ \in \mathbb{C} : \quad \neq 0 \text{ and } |\arg\ | < \omega \Big\}.$$

Assume that

(1) is holomorphic on S_ω;

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(2) is bounded near the origin;

(3) $|\ (\ )| \le C|\ |$ for $|\ | > 1$.

We consider the function:

$$H\ (\ ,\bar{\xi}) = \sum_{=1}^{\infty}\ (\ )\sum_{=1}^{N}\ (\ )\ \overline{\ (\xi)}. \tag{1.2}$$

If $(\ ) \equiv 1$, then (1.2) becomes (1.1). For $= 0$, Cowling-Qian [1] introduced a class of bounded holomorphic multipliers on $L^2(\partial\mathbb{B}\ )$. In this paper, we consider the case $\ne 0$. For this case, is unbounded on $\{\ :|\ |>1\}$. We prove that if $\in H\ (S_{\omega})$, then

$$|H\ (\ ,\bar{\xi})| = \frac{C_{\mu'}}{\delta(\nu,\mu')|1-\ \bar{\xi}'|^{\ +}}.$$

See Theorem 3.4.

In Section 4, we introduce a class of Fourier multipliers M with $\in H\ (S_{\omega})$, $\ne 0$. Unlike the ones of Cowling-Qian [1], our multipliers are unbounded on S_{ω}. Take $(\ ) =$. Plancherel's theorem implies that M is not bounded on $L^2(\partial\mathbb{B}\ )$. Hence for such M , we need to consider their boundedness on some function spaces with higher regularity. Let $,\ \in [0,\infty)$. We prove that if $\in H\ (S_{\omega})$, M is bounded from Sobolev space $^{\ ,\ +}(\partial\mathbb{B}\ )$ to Sobolev space $^{\ ,}(\partial\mathbb{B}\ )$, $1 < \ < \infty$. Our result implies that the operators M take a role of fractional differential operators on $\partial\mathbb{B}$. See Theorem 4.5.

The rest of this paper is organized as follows. In Section 2, we state some basic preliminaries and notations which will be used in the sequel. In Section 3, we estimate the kernels generated by holomorphic multipliers $\in H\ (S_{\omega})$. The Sobolev boundedness of the operators M is given in Section 4.

N : $\mathsf{U} \approx \mathsf{V}$ represents that there is a constant > 0 such that $^{-1}\mathsf{V} \le \mathsf{U} \le \mathsf{V}$ whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, one writes $\mathsf{V} \gtrsim \mathsf{U}$ for $\mathsf{V} \ge \ \mathsf{U}$.

2. Preliminaries and notations

In this section we state some preliminaries and notations and refer the reader to Gong [4], Hua [5] and Rudin [13] for further information. We use as a general element of $\mathbb{C}$, i.e. $= (\ _1,\cdots,\ )$, $\in \mathbb{C}$, $= 1,2,\cdots,\ ,\ \ge 2$. Denote $\bar{\ } = [\overline{\ _1},\cdots\overline{\ }]$. The notation is considered to be a row vector. Denote by $\mathbb{B}$ the open unit ball $\{\ \in\mathbb{C}\ :|\ |<1\}$, where $|\ | = \left(\sum_{=1}|\ |^2\right)^{1/2}$. The unit sphere in $\mathbb{C}$ is denoted by

$$\partial\mathbb{B}\ = \mathbb{S}^{2\ -1} = \left\{\ \in\mathbb{C}\ :|\ |=1\right\}.$$

The open ball centered at with radius will be denoted by $B(\ ,\ )$. A general element on $\partial\mathbb{B}$ is usually denoted by ξ. The constant $\omega_{2\ -1}$ involved in the Cauchy-Szegö kernel is the surface area of $\partial\mathbb{B}$ and is equal to $\frac{2\pi}{\Gamma(\ )}$. For $,\ \in\mathbb{C}$, we use the notation $\ ' = \sum_{=1}$. The theory developed in this paper is relevant to the radial Dirac operator

$$D = \sum_{=1} \frac{\partial}{\partial} .$$

Now we state some basis knowledge of basis functions in the space of holomorphic function in $\mathbb{B}$ and some relevant function spaces on $\partial\mathbb{B}$. We refer to Hua [5] for details. Let be a nonnegative integer. We consider the colum vector $^{[\ ]}$ with components

$$\sqrt{\frac{!}{{}_1! \cdots \ !}}\ {}_1^{\ 1} \cdots \quad , \quad {}_1 + \cdots \quad = \quad .$$

The dimension of $^{[\ ]}$ is

$$N \ = \frac{1}{!} \ (\ + 1) \cdot (\ + \ \ - 1) = C_{\ + \ -1}.$$

Let and $\sigma(\xi)$ be the Lebesgue volume element of $\mathbb{C}$ and the Lebesgue area element of $\partial\mathbb{B}$, respectively. Define

$$\begin{cases} H_1 = \displaystyle\int_{\mathbb{B}} \overline{\ {}^{[\ ]\prime}} \cdot \ {}^{[\ ]} \ , \\ H_2 = \displaystyle\int_{\partial\mathbb{B}} \overline{\xi^{[\ ]\prime}} \cdot \xi^{[\ ]} \sigma(\xi). \end{cases}$$

It is easy to prove that H_1 and H_2 are positive definite Hermitian matrices of order N . There exists a matrix Γ such that

$$\begin{cases} \overline{\Gamma'} \cdot H_1 \cdot \Gamma = \Lambda, \\ \overline{\Gamma'} \cdot H_2 \cdot \Gamma = I, \end{cases} \tag{2.1}$$

where $\Lambda = [\beta_1, \cdots, \beta \ ]$ is a diagonal matrix and I is the identity matrix. Set

$$\begin{cases} \ {}_{[\ ]} = \ {}^{[\ ]} \cdot \Gamma; \\ \xi_{[\ ]} = \xi^{[\ ]} \cdot \Gamma. \end{cases}$$

Denote by $(\ )$ the components of the vectors $_{[\ ]}$. From (2.1), we can see that

$$\int_{\mathbb{B}} \ (\ ) \overline{\ {}_\mu(\ )} \ = \delta \ {}_\mu \cdot \delta \ \ \cdot \beta \ , \tag{2.2}$$

$$\int_{\partial\mathbb{B}} \ (\xi) \overline{\ {}_\mu(\xi)} \sigma(\xi) = \delta \ {}_\mu \cdot \delta \ \ . \tag{2.3}$$

The following theorem is well known.

Theorem 2.1. *T*

$$\left\{ (\beta_\nu)^{-\frac{1}{2}} \ {}_\nu, \quad = 0, 1, 2, \cdots, \nu = 1, 2, \cdots, N \ \right\}$$

$\mathbb{B}$. *T*

$\{ \ {}_\nu \}$,

$\partial\mathbb{B}$.

The explicit formula of the Cauchy-Szegö kernel

$$H(\ ,\bar{\xi}) = \frac{1}{\omega_{2\ -1}} \frac{1}{(1 - \ \overline{\xi}')}$$

on $\partial\mathbb{B}$ was first deduced in Hua [5] by using the system $\{\ \}$ and the relation

$$H(\ ,\bar{\xi}) = \sum_{=0}^{\infty} \sum_{=1}^{N} (\ ) \overline{\ (\xi)}, \quad \in \mathbb{B}\ , \xi \in \mathbb{B}\ . \tag{2.4}$$

For $\ , \omega \in \mathbb{B} \bigcup \partial\mathbb{B}$, the nonisotropic distance $(\ ,\omega)$ is defined as

$$(\ ,\omega) = \left|1 - \ \bar{\omega}'\right|^{1/2}.$$

It can be easily shown that $(\cdot,\cdot)$ is a metric on $\partial\mathbb{B}$. For $\xi \in \partial\mathbb{B}$ and $\varepsilon > 0$, we define the ball corresponding to $(\cdot,\cdot)$ as

$$S(\xi,\varepsilon) = \left\{\eta \in \partial\mathbb{B}\ , (\xi,\eta) \leq \varepsilon\right\}.$$

The complement set of $S(\xi,\varepsilon)$ in $\partial\mathbb{B}$ is denoted by $S\ (\xi,\varepsilon)$.

Set

$$\mathcal{A} = \left\{\ :\ \text{ is holomorphic in } B(0, 1+\delta) \text{ for some } \delta > 0\right\}.$$

If $\ \in \mathcal{A}$, then

$$(\ ) = \sum_{=0}^{\infty} \sum_{=0}^{N} \ (\ ),$$

where are the Fourier coefficients of :

$$= \int_{\partial\mathbb{B}} \overline{\ (\xi)}\ (\xi) d\sigma(\xi),$$

and for any positive integer , the series

$$\sum_{=0}^{\infty} \sum_{=0}^{N} \ (\ )$$

is uniformally and absolutely convergent in any compact ball contained in $B(0, 1 + \delta)$ in which is defined.

Denote by $\mathcal{U}$ the unitary group of $\mathbb{C}$ consisting of all unitary operators on the Hilbert space $\mathbb{C}$ under the complex inner product $\langle\ ,\ \rangle = \ \overline{\ }'$. These are the linear operators that preserve inner products:

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle.$$

Clearly, $\mathcal{U}$ is a compact subset of $O(2\ )$. It is easy to verify that $\mathcal{A}$ is invariant under $\ \in \mathcal{U}$. If $\ \in \mathcal{A}$, then is defined by its values on $\partial\mathbb{B}$. In Section 3, we treat $|_{\partial\mathbb{B}}$ as identical to $\ \in \mathcal{A}$.

3. The kernel generated by holomorphic multipliers

Set

$$\begin{aligned} S_\omega &= \Big\{ \ \in \mathbb{C} \mid \ \neq 0 \text{ and } |\arg \ | < \omega \Big\}, \\ S_\omega(\pi) &= \Big\{ \ \in \mathbb{C} \mid \ \neq 0, |\mathrm{Re}(\ )| \leq \pi \text{ and } |\arg(\pm \ )| < \omega \Big\}, \\ {}_\omega(\pi) &= \Big\{ \ \in \mathbb{C} \mid \ \neq 0, |\mathrm{Re}(\ )| \leq \pi \text{ and } \mathrm{Im}(\ ) > 0 \Big\} \bigcup S_\omega(\pi), \\ H_\omega &= \Big\{ \ \in \mathbb{C} \mid \ = \ ^{\omega}, \omega \in \ {}_\omega(\pi) \Big\}. \end{aligned}$$

The following function space is relevant:

Definition 3.1. Let $-1 < \ < \infty$. $H \ (S_\omega)$ is defined as the set of all holomorphic functions in S_ω such that

(1) is bounded for $| \ | \leq 1$;
(2) $| \ (\ )| \leq C_\mu | \ | \ , \ \in S_\mu, 0 < \mu < \omega$.

R 3.2. The classes $H \ (S_\omega)$ are generalizations of $H^\infty(S_\omega)$ which is introduced by A. McIntosh and his collaborators. We refer to Li-McIntosh-Semmes [6], McIntosh [7], McIntosh-Qian [8], Qian [12] and the reference therein for further information on $H^\infty(S_\omega)$.

Let

$$\varphi \ (\ ) = \sum_{=1}^{\infty} \ (\ ) \ .$$

Lemma 3.3. *L* $\in H \ (S_\omega)$, $-1 < \ < \infty$. *T* φ H_ω. *M* , $0 < \mu < \mu' < \omega$ $= 0, 1, 2, \ldots,$

$$\left| \left(\frac{\mathrm{d}}{\mathrm{d}} \right) \varphi \ (\ ) \right| \lesssim \frac{C_{\mu'} \ !}{\delta \ (\mu, \mu') |1 - \ |^{\ +1+}}, \quad \in H_\mu,$$

$\delta(\mu, \mu') = \min\{\frac{1}{2}, \tan(\mu, \mu'\}$; $C_{\mu'}$ *D fi* *3.1.*

P . Let

$$\begin{aligned} {}_\omega &= \Big\{ \ \in \mathbb{C} : \mathrm{Im}(\ ) > 0 \Big\} \bigcup S_\omega \bigcup (-S_\omega), \\ {}_\omega &= \ {}_\omega \cap \Big\{ \ \in \mathbb{C} : \ -\pi \leq \mathrm{Re} \ \leq \pi \Big\} \end{aligned}$$

and ρ_θ is the ray $\exp(\ \theta), 0 < \ < \infty$, where θ is chosen so that $\rho_\theta \subsetneq S_\omega$. Define

$$\Psi \ (\ ) = \frac{1}{2\pi} \int_{\rho(\theta)} \exp(\ \xi \ ) \ (\xi) \mathrm{d}\xi, \quad \in \ {}_\omega,$$

where $\exp(\ \xi)$ is exponentially decaying as $\xi \to \infty$ along ρ_θ. Then we get

$$\begin{aligned} \left| | \ |^{1+} \ \Psi \ (\ ) \right| &= \left| \frac{1}{2\pi} \int_{\rho(\theta)} \exp(\ \xi \ ) | \ |^{1+} \ (\xi) \mathrm{d} \ \right| \\ &\lesssim \frac{C_{\mu'}}{2\pi} \int_0^\infty \exp(- \ | \ | \sin(\theta + \arg \ ))(\ | \ |) \ \mathrm{d}(\ | \ |) \\ &\lesssim C_{\mu'}, \end{aligned} \tag{3.1}$$

which implies $|\Psi\ (\ )| \lesssim 1/|\ |^{1+}$. Define

$$\psi\ (\ ) = 2\pi \sum_{=-\infty}^{\infty} \Psi\ (\ \ +2\ \ \pi), \qquad \in \bigcup_{=-\infty}^{\infty} (2\ \ \pi + \ \ _{\omega}).$$

It is easy to see that ψ is holomorphically and 2π -periodically defined in the described region, and $|\psi\ (\ )| \lesssim 1/|\ |^{1+}$. Let

$$\varphi\ (\ ) \quad = \quad \psi\ \left(\frac{\log}{\ }\right).$$

For $\ \in$

P . Recall that

$$\begin{cases} \varphi\ (\ ) = \sum_{=1}^{\infty}\ (\ )\ ; \\ {}^{-1}\varphi\ (\ ) = \sum_{=1}^{\infty}\ (\ )\ {}^{+\ -1}. \end{cases}$$

Then we have

$$\begin{aligned} \frac{1}{(\ -1)!}\Big(\ {}^{-1}\varphi\ (\ )\Big)^{(\ -1)} &= \frac{1}{(\ -1)!}\sum_{=1}^{\infty}\ (\ )(\ +\ \ -1)(\ +\ \ -2)\ldots(\ +1) \\ &= \sum_{=1}^{\infty}\ (\ )\ \frac{(\ +\ \ -1)!}{(\ -1)!\ !} \\ &= \sum_{=1}^{\infty}\frac{(\ +\ \ -1)(\ +\ \ -2)(\ +1)}{!}\ (\ )\ , \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{(\ -1)!}\Big(\ {}^{-1}\varphi\ (\ )\Big)^{(\ -1)}\Big|_{=\bar{\xi}'} &= \sum_{=1}^{\infty}\ (\ )\frac{(\ +\ \ -1)(\ +\ \ -2)(\ +1)}{!}(\ \bar{\xi}') \\ &= \omega_{2\ -1}\sum_{=1}^{\infty}\ (\ )\sum_{=1}^{N}\ (\ )\ \overline{(\xi)} \\ &= \omega_{2\ -1}H\ (\ ,\bar{\xi}). \end{aligned}$$

□

By [10, Theorem 3], we could obtain the following result.

Theorem 3.5. *L* . *I* $\in H\ (S_{\omega,\pm})$,

$$H\ (\ ,\xi) = \sum_{=1}^{\infty}\ (\ )\sum_{=1}^{N}\ (\ )\ {}_{\mu}(\xi),\quad \in \mathbb{B},\ \xi \in \partial\mathbb{B}\ ,$$

$$\left|D\ H\ (\ ,\bar{\xi})\right| \lesssim \frac{C_\mu\ !\left[|\ln|1-\ \ \bar{\xi}'||+1\right]}{\delta\ (\mu,\mu')|1-\ \ \bar{\xi}'|^{\ +\ +}}.$$

P . The proof is similar to Theorem 3.4. we omit it. □

4. Sobolev spaces and unbounded Fourier multipliers

4.1. **Integral representation of multipliers.** Given $\in H\ (S_\omega)$. We define an Fourier multiplier operator $M\ : \mathcal{A} \to \mathcal{A}$ by

$$M\ (\ )(\xi) = \sum_{=1}^{\infty}\ (\ )\sum_{=0}^{N}\ \ (\xi),\ \xi \in \partial\mathbb{B}\ ,$$

where { } are the Fourier coefficients of the test function $\in \mathcal{A}$.

For the above operator M , a Plemelj type formula holds.

Theorem 4.1. *L* $\in H\ (S_\omega)$, > 0. *T* ${}_1(\ ) = {}^{-1}\ (\ )$, ${}_1 = [\ ] + 1$. *O* *M* . *F* $\in \mathcal{A}$,

$$
\begin{aligned}
M\ (\ )(\xi) &= \lim_{\varepsilon\to 0}\Big[\int_{S\ (\xi,\varepsilon)} H\ {}_1(\xi,\overline{\eta})D_\eta^{1}\ (\eta)\ \sigma(\eta)\\
&\quad +(D^{1}\ )(\xi)\int_{S\ (\xi,\varepsilon)} H\ {}_1(\xi,\overline{\eta})\ \sigma(\eta)\Big],
\end{aligned}
$$

$\int_{S(\xi,\varepsilon)} H\ {}_1(\xi,\overline{\eta})\ \sigma(\eta)$ $\xi \in \partial\mathbb{B}$ ε.

P . Let

$$
M\ (\ )(\rho\xi) = \sum_{=1}^{\infty}\ (\ )\sum_{=1}^{N}\ (\rho\xi),\quad \xi\in\partial\mathbb{B}\ ,
$$

where

$$
= \int_{\partial B}\overline{\ (\eta)}\ (\eta)\mathrm{d}\sigma(\eta).
$$

We can see that

$$
\begin{aligned}
D^{\ [\ ]} &= \sqrt{\frac{!}{{}_1!\,{}_2!\cdots\ !}}\sum_{=1}\ \frac{\partial}{\partial}\Big({}_1^{1}\ {}_2^{2}\cdots\ \Big)\\
&= \sqrt{\frac{!}{{}_1!\,{}_2!\cdots\ !}}\sum_{=1}\ {}_1^{1}\ {}_2^{2}\cdots\ {}_{-1}^{-1}\ {}^{-1}\ {}_{+1}^{+1}\cdots\\
&= \sqrt{\frac{!}{{}_1!\,{}_2!\cdots\ !}}\Big(\sum_{=1}\ \Big)\ {}_1^{1}\ {}_2^{2}\cdots\\
&= {}^{[\ ]},
\end{aligned}
$$

which implies that $D\ =\ $. Then we have

$$
\begin{aligned}
M\ (\ )(\rho\xi) &= \sum_{=1}^{\infty}\ (\ )\sum_{=1}^{N}\int_{\partial B}\ (\rho\xi)\overline{\ (\eta)}\ (\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{=1}^{\infty}\ (\ )\frac{1}{{}_1}\sum_{=1}^{N}\int_{\partial B}\ (\rho\xi)\ {}^{1}\overline{\ (\eta)}\ (\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{=1}^{\infty}\ (\ )\frac{1}{{}_1}\sum_{=1}^{N}\int_{\partial B}\ (\rho\xi)D_\eta^{1}\overline{\ (\eta)}\ (\eta)\mathrm{d}\sigma(\eta).
\end{aligned}
$$

By integration by parts,

$$
\begin{aligned}
M\ (\ )(\rho\xi) &= \sum_{=1}^{\infty}\ (\ )\frac{1}{{}_1}\sum_{=1}^{N}\int_{\partial B}\ (\rho\xi)\overline{\ (\eta)}(D_\eta^{1}\ )(\eta)\mathrm{d}\sigma(\eta)\\
&= \sum_{=1}^{\infty}\ {}_1(\ )\ \sum_{=1}\ {}_1(\ \psi\psi(\eta)(D_\eta\psi\psi)(\eta)\mathrm{d}\sigma(\eta)\psi\psi\psi\eta\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi\psi
\end{aligned}
$$

For any $\varepsilon > 0$, we have

$$\begin{aligned} M\ (\ )(\rho\xi) &= \int_{S\ \ (\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta}) D_{\eta}^{\ 1}\ \ (\eta)\mathrm{d}\sigma(\eta) \\ &+ \int_{S(\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta})(-D_{\xi}^{\ 1}\ \ (\xi) + D_{\eta}^{\ 1}\ \ (\eta))\mathrm{d}\sigma(\eta) \\ &+ D_{\xi}^{\ 1}\ \ (\xi)\int_{S(\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta) \\ &=: I_1(\rho,\varepsilon) + I_2(\rho,\varepsilon) + D_{\xi}^{\ 1}\ \ (\xi) I_3(\rho,\varepsilon), \end{aligned}$$

where

$$\begin{aligned} I_1(\rho,\varepsilon) &= \int_{S\ \ (\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta}) D_{\eta}^{\ 1}\ \ (\eta)\mathrm{d}\sigma(\eta), \\ I_2(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta})(-D_{\xi}^{\ 1}\ \ (\xi) + D_{\eta}^{\ 1}\ \ (\eta))\mathrm{d}\sigma(\eta), \\ I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta). \end{aligned}$$

For $\rho \to 1-0$, we have

$$\begin{aligned} \lim_{\rho\to 1-0} I_1(\rho,\varepsilon) &= \lim_{\rho\to 1-0}\int_{S\ \ (\xi,\varepsilon)} H_{\ 1}(\rho\xi,\bar{\eta}) D_{\eta}^{\ 1}\ \ (\eta)\mathrm{d}\sigma(\eta) \\ &= \int_{S\ \ (\xi,\varepsilon)} H_{\ 1}(\xi,\bar{\eta}) D_{\eta}^{\ 1}\ \ (\eta)\mathrm{d}\sigma(\eta). \end{aligned}$$

Now we consider $I_2(\rho,\varepsilon)$. Let $\xi = [1,0,\ldots,0]$. For $\eta \in \partial\mathbb{B}\ $, write

$$\begin{cases} \eta_1 = \ \ ^{\theta}, \eta_2 = \ \ _2, \eta_3 = \ \ _3, \ldots, \eta\ = \ \ ; \\ \ = [\ _2,\ \ _3, \ldots,\ \ ]. \end{cases}$$

For such $\eta \in \partial\mathbb{B}\ $, $\ \bar{}\ ' = 1 - \ ^2$. Without loss of generality, assume $\xi = 1$. We get

$$\left|1-\xi\bar{\eta}'\right|^{1/2} = \left|1 - \ \ ^{\theta}\right|^{1/2} = [(1-\ \ \cos\theta)^2 + (\ \ \sin\theta)^2]^{1/4} \le \varepsilon,$$

which implies that

$$\cos\theta \ge \frac{1+\ ^2-\varepsilon^4}{2}.$$

The above estimate implies

$$S(\xi,\varepsilon) = \left\{\eta \mid \ \bar{}\ ' = 1 - \ ^2, \cos\theta \ge \frac{1+\ ^2-\varepsilon^4}{2}\right\}.$$

Since

$$\frac{1+\ ^2-\varepsilon^4}{2} \le \cos\theta \le 1,$$

we obtain $1 - \ \ \le \varepsilon^2$ and then

$$\bar{}\ ' = 1 - \ ^2 \le 1-(1-\varepsilon^2)^2 = 2\varepsilon^2 - \varepsilon^4.$$

Denote

$$= \ (\ , \varepsilon) = \arccos\left(\frac{1 + \ ^2 - \varepsilon^4}{2}\right).$$

Since $(1 - \ )^2 \leq \varepsilon^4$ and $1 - \ = O(\arccos^2\ )$, we get $\ = O(\varepsilon^2)$. It is easy to see

$$\begin{aligned} |\xi - \eta|^2 &= |1 - \ ^{\theta}|^2 + \sum_{=2} |\ |^2 \\ &= (1 + \ ^2 - 2\ \cos\theta) + (1 - \ ^2) \\ &= 2 - 2\ \cos\theta \end{aligned}$$

and

$$\begin{aligned} d^4(\xi, \eta) &= 1 + \ ^2 - 2\ \cos\theta \\ &= (2 - 2\ \cos\theta) - (1 - \ ^2) \\ &= |\xi - \eta|^2 - (1 + \ )(1 - \ ), \end{aligned}$$

that is, $d^2(\xi, \eta) \leq |\xi - \eta|$. Because

$$d^2(\xi, \eta) = [1 + \ ^2 - 2\ \cos\theta]^{1/2} \geq 1 - \ ,$$

then we have $1 - \ \leq d^2(\xi, \eta)$, so

$$|\xi - \eta|^2 \leq d^4(\xi, \eta) + (1 + \ )d^2(\xi, \eta).$$

Since $d^2(\xi, \eta) \leq 2$, then

$$|\xi - \eta|^2 \leq 2d^2(\xi, \eta) + 2d^2(\xi, \eta) = 4d^2(\xi, \eta),$$

that is

$$|\xi - \eta| \leq 2d(\xi, \eta).$$

Since $\ \in \mathcal{A}$, we have

$$|\ (\xi) - \ (\eta)| \lesssim |\xi - \eta| \lesssim d(\xi, \eta).$$

For $\rho \in (0, 1)$

$$\begin{aligned} |I_2(\rho, \varepsilon)| &\lesssim \int_{S(\xi,\varepsilon)} \left|H_{\ 1}(\rho\xi, \bar{\eta})\right| |\ (\xi) - \ (\eta)|\, \mathrm{d}\sigma(\eta) \\ &\lesssim \int_{S(\xi,\varepsilon)} \frac{d(\xi, \eta)}{|1 - \xi\bar{\eta}'|} \mathrm{d}\sigma(\eta) \\ &\lesssim \int_{-'\leq 2\varepsilon^2 - \varepsilon^4} \int_{-} \frac{1}{\left|1 - \ ^{\theta}\right|^{\ -1/2}} \mathrm{d}\theta \mathrm{d}\ . \end{aligned}$$

For $\;= 2$,

$$\begin{aligned}
\frac{1}{2}\int_{-} \frac{1}{\left|1 - \quad {}^{\theta}\right|^{2-1/2}}\mathrm{d}\theta &\leq \left(\frac{1}{2}\int_{-} \frac{1}{\left|1 - \quad {}^{\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4} \\
&\leq \left(\frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{\left|1 - \quad {}^{\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4} \\
&\leq \left(\frac{1}{2}\right)^{3/4} \frac{1}{(1 - \quad {}^{2})^{3/4}}.
\end{aligned}$$

Then we get

$$\begin{aligned}
|I_2(\rho,\varepsilon)| &\lesssim \int_{-' \leq 2\varepsilon^2-\varepsilon^4} \quad {}^{1/4}\frac{1}{(1 - \quad {}^{2})^{3/4}}\mathrm{d} \\
&\lesssim \varepsilon^{1/2}\int_{-' \leq 2\varepsilon^2-\varepsilon^4} \frac{1}{(\quad -')^{3/4}}\mathrm{d} \\
&= \varepsilon^{1/2}\int_{0}^{\sqrt{2\varepsilon^2-\varepsilon^4}} \frac{}{\;^{3/2}}\mathrm{d} \\
&\lesssim \varepsilon \to 0
\end{aligned}$$

For $\;> 2$, we have

$$\begin{aligned}
\int_{-} \frac{1}{\left|1 - \quad {}^{\theta}\right|^{\; -1/2}}\mathrm{d}\theta &\lesssim \int_{-} \frac{\left|1 - \quad {}^{2}\right|^{\; -1/2-2}}{\left|1 - \quad {}^{\theta}\right|^{\; -1/2}}\frac{1}{\left|1 - \quad {}^{2}\right|^{\; -1/2-2}}\mathrm{d}\theta \\
&\lesssim \frac{1}{\left|1 - \quad {}^{2}\right|^{\; -1/2-1}}\int_{-\pi}^{\pi} \frac{1}{\left|1 - \quad {}^{\theta}\right|^{2}}\mathrm{d}\theta \\
&\lesssim \frac{1}{\left|1 - \quad {}^{2}\right|^{\; -1/2-1}},
\end{aligned}$$

then we get

$$|I_2(\rho,\varepsilon)| \lesssim \int_{0}^{\sqrt{2\varepsilon^2-\varepsilon^4}} \quad {}^{2\; -3}\frac{1}{\;^{2\; -3}}\mathrm{d} \lesssim \sqrt{2\varepsilon^2} \to 0.$$

Now we prove if $\rho \to 1-0$, $I_3(\rho,\varepsilon)$ has a limit uniformly bounded for ε near 0. Integrating as before, we have

$$\begin{aligned}
I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{\;1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta) \\
&= \int_{-' \leq 2\varepsilon^2-\varepsilon^4}\int_{-} \left(\quad^{-1}\varphi_{\;1}(\;)\right)^{(\; -1)}\Big|_{=\rho \quad \theta} \mathrm{d}\theta\mathrm{d}\; .
\end{aligned}$$

Let $\;= \rho \quad {}^{\theta}$. Then $\mathrm{d}\; = \quad \mathrm{d}\theta$. We get

$$I_3(\rho,\varepsilon) = -\int_{-' \leq 2\varepsilon^2-\varepsilon^4}\int_{\rho \quad -}^{\rho} \left(\quad^{-1}\varphi_{\;1}(\;)\right)^{(\; -1)}\mathrm{d}\;\mathrm{d}\; .$$

By integration by parts, the inside integral with respect to the variable becomes

$$\int_{-} \left({}^{-1}\varphi_{1}( ) \right)^{(-1)} \Big|_{-1}$$

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On the other hand

$$(\; - 1)! \int_{\rho}^{\rho} {}_{-} \; \frac{\varphi \; {}_{1}(\;)}{\;} \mathrm{d} \; = \; (\; - 1)! \int_{-} \; \varphi \; {}_{1}(\;)\Big|_{=\rho} \; {}_{\theta} \, \mathrm{d}\theta$$

$$\lesssim \; 1, \; (\text{ when } \rho \to 0)$$

that implies

$$\int_{-' \leq 2\varepsilon^2 - \varepsilon^4} L(\rho \; , \;)\mathrm{d} \; .$$

□

4.2. **Sobolev spaces on $\partial\mathbb{B}$ via Fourier mulitpliers.** Sobolev spaces on the - complex unit sphere $\partial\mathbb{B}$ are defined as follows. We define the fractional integral operator $\mathcal{I}$ on $\partial\mathbb{B}$ as follows. Let

$$(\;) = \sum_{=0}^{\infty} \sum_{=0}^{N} \quad (\;).$$

For $-\infty < \quad < \infty$, the operator $\mathcal{I}$ is defined by

$$\mathcal{I} \quad (\;) = \sum_{=0}^{\infty} \sum_{=0}^{N} \quad (\;).$$

For $\in \mathbb{Z}_{+}$, we can see that the operators $\mathcal{I}$ become the ordinary differential operators with higher orders.

Theorem 4.2. *L* $\in \mathbb{Z}_{+}$. $D \; = \mathcal{I}$ $L^2(\partial\mathbb{B} \;)$.

P . Without loss of generality, we assume that $\in \mathcal{A}$. Then

$$(\;) = \sum_{=0}^{\infty} \sum_{=0}^{N} \quad (\;),$$

where are the Fourier coefficients of :

$$= \int_{\partial\mathbb{B}} \overline{\quad (\xi)} \; (\xi) \mathrm{d}\sigma(\xi).$$

So

$$\begin{aligned} D \quad (\;) &= \sum_{=0}^{\infty} \sum_{=0}^{N} \int_{\partial\mathbb{B}} \overline{\quad (\xi)} \; (\xi) \mathrm{d}\sigma(\xi) D \; (\quad)(\;) \\ &= \sum_{=0}^{\infty} \quad \sum_{=0}^{N} \int_{\partial\mathbb{B}} \overline{\quad (\xi)} \; (\xi) \mathrm{d}\sigma(\xi) \quad (\;). \end{aligned}$$

□

Definition 4.3. Let $\in [0, +\infty)$. The Sobolev norm $\| \cdot \|_{\;2,\;(\partial\mathbb{B}\;)}$ on $\partial\mathbb{B}$ is defined as

$$\| \; \|_{\;2,\;(\partial\mathbb{B}\;)} =: \|\mathcal{I} \quad \|_2 < \infty.$$

The Sobolev spaces on $\partial\mathbb{B}$ is defined as the closure of $\mathcal{A}$ under the norm $\| \cdot \|_{\;2,\;(\partial\mathbb{B}\;)}$, that is $^{2,}(\partial\mathbb{B} \;) = \overline{\mathcal{A}}^{\|\cdot\|_{\;2,\;(\partial\mathbb{B}\;)}}$.

R 4.4.

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