Rational Orthogonal Systems Are Schauder Bases

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Abstract

It is well known that orthogonal rational systems (Takenaka-Malmquist or TM systems) are bases of the closures of their spans in all Hardy H^p spaces, $1 < p < \infty$. In this paper we further prove that they are, in fact, Schauder bases in those spaces. We simultaneously treat both the contexts the unit disc $\mathbf{D}$ and the upper-half space $\mathbf{C}^+$.

Keywords: Schauder Bases, Hilbert transform, nonlinear phase, Hardy space

1 Introduction

In a Banach space B a set E is called a basis [1] of B if it satisfies $\overline{\text{span}}\{\mathsf{E}\} = \mathsf{B}$, where $\text{span}\{\mathsf{E}\}$ stands for the collection of all finite linear combinations of elements in E, $\overline{\text{span}}$ is for the topological closure. In a Banach space B a collection of elements $\{e_n\}$ is said to be a Schauder basis if it is first a basis and, secondly,

$$\lim_{n\to\infty} \|f - s_n(f)\| = 0,$$

where

$$s_n(f) = \sum_{k=1}^{n} \Lambda_k(f) e_k$$

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is the partial sum and $\{\Lambda_n\}$ is a sequence of bounded linear functionals of $\{E\}$. It is well known that in a Hilbert space any basis is a Schauder basis.

It is well known that the Fourier system [2, 3] $\{\frac{e^{ikt}}{\sqrt{2\pi}}\}_{k=-\infty}^{\infty}$ is a Schauder basis in $L^p([-\pi,\pi])$ for all $1 < p < \infty$; and so is the half-Fourier system $\{\frac{e^{ikt}}{\sqrt{2\pi}}\}_{k=0}^{\infty}$ in the Hardy spaces $H^p(\partial D)$, $1 < p < \infty$. The Hardy spaces [4] on the boundary consist of the non-tangential boundary limits of the related holomorphic Hardy space functions inside the open unit disc. For the same p, the two types of Hardy spaces are isometry. The same is for the upper-half complex plane context.

The TM systems [5], including the Laguerre and the Kautz systems [6], are natural generalizations of the half-Fourier system. They enjoys a long term development with ample applications in both pure [7, 8] and applied mathematics, including control theory [9, 10].

There are usually two different contexts, viz., the unit disc and the upper- half complex plane. In the unit disc case, for a given parameter sequence $\{a_n\}_{n=1}^{\infty} \subset D$, the corresponding TM system $\{e_n\}_{n=1}^{\infty}$ is

$$e_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{j=1}^{n-1}\frac{z-a_j}{1-\overline{a_j}z}, \quad z \in D, \quad n = 1, 2, \ldots, \tag{1.1}$$

where for $n = 1$ we use the convention $\prod_{j=1}^{0} = 1$.

Similarly, for a given parameter sequence $\{\lambda_n\}_{n=1}^{\infty} \subset C^+$, the corresponding TM system $\{\beta\}_{n=1}^{\infty}$ is

$$\beta_n(w) = \frac{\sqrt{\frac{2}{\pi}Im\{\lambda_n\}}}{w-\overline{\lambda_n}}\prod_{j=1}^{n-1}\frac{w-\lambda_j}{w-\overline{\lambda}_j}, \quad w \in C^+, \quad n = 1, 2, \ldots \tag{1.2}$$

The case $a_n = 0, n = 1, 2, \cdots$, corresponds to the half-Fourier system that gives rise to classical Fourier analysis. It is an important result in classical Fourier analysis that in all $L^p(\partial D)$ ($H^p(\partial D)$), $1 < p < \infty$, the Fourier system (the half-Fourier system) is a Schauder basis (see, for instance, [2]). It is natural to ask whether a general TM system is a Schauder basis in the closure of its span in those Hardy spaces. To the authors' knowledge this question has been open. In this paper we give the positive answer to this question. In each of the mentioned two contexts there are two different cases to make. It is according to whether or not the parameter sequence giving rise to the system can define a Blaschke product in the context. In the unit disc context, the two cases are

$$\sum_n (1-|a_n|) < \infty \quad \text{or} \quad \sum_n (1-|a_n|) = \infty. \tag{1.3}$$

In the first case the parameter sequence $\{a_n\}$ define a Blaschke product: The Blaschke product has the parameters in the sequence as all its zeros together with the multiples. In the second case the parameters cannot define a Blaschke product. In the second case the closure of the span of $\{e_n\}$ is the whole Hardy space $H^p(\partial D)$, and in the first case not. Similarly, in the upper-half space context we have

$$\sum_{k=1}^{\infty}\frac{\sqrt{Im(\lambda_k)}}{1+|\lambda_k|^2} < \infty \quad \text{or} \quad \sum_{k=1}^{\infty}\frac{\sqrt{Im(\lambda_k)}}{1+|\lambda_k|^2} = \infty. \tag{1.4}$$

The second case corresponds to the closure of the span $\{\beta_n\}$ being the whole space, but the first case not. The result in the unit disc context is well known. The proofs can be found, for instance, in [1] or [6]. The corresponding result in the upper-half plane context is also known. There have been increasing interests in the applications of TM system including control theory, system identification and signal theory[6, 11, 12]. An extension of it in relation to general backward shift invariant subspaces (see below) is given in [12]. This paper does not depend on the results for general p, but only the case $p = 2$ in which a basis is at the same time a Schauder basis. In below when a result is valid for both the unit disc and the upper-half plane contexts we suppress the notation specifying the region, viz., $\mathbf{D}$ and $\mathbf{C}^+$, and write the basis as $\{\alpha_n\}$. Based on this convention, corresponding to $p = 2$, the following is understood to be valid for both contexts:

$$H^2 = \overline{\mathrm{span}}^2\{\alpha_n\} \bigoplus B\overline{H^2}, \tag{1.5}$$

where, here and below, $\overline{\mathrm{span}}^p$ is for the H^p closure, B is the Blaschke product in either of the two contexts defined by the related parameter sequence ([11]). The above result implies that if the parameter sequence cannot define a Blaschke product, then

$$H^2 = \overline{\mathrm{span}}^2\{\alpha_n\}. \tag{1.6}$$

No matter whether or not a parameter sequence is qualified to define a Blaschke product, we always have

$$\overline{\mathrm{span}}^p\{\alpha_n\} = H^p \cap B\overline{H^p} = (BH^{p'})^{\perp}, \tag{1.7}$$

where the notation $\perp$ is for the formal orthogonality under the paring between H^p and $H^{p'}$, $1/p + 1/p' = 1, 1 < p, p' < \infty$. Note that subspaces on the right-hand-side, $H^p \cap B\overline{H^p}$, in either of the two contexts, represent the backward-shift-invariant spaces of the H^p ([4]). In the unit disc context the shift operator and the backward shift operators are, respectively,

$$S^* f(z) = zf(z), \quad Sf(z) = \frac{f(z) - f(0)}{z}, \quad z \in \mathbf{D};$$

and in the upper-half plane, by using the same notation, they are

$$S^* f(\zeta) = e^{ia\zeta} f(\zeta), \quad Sf(\zeta) = e^{-ia\zeta} f(\zeta), \quad a > 0, \ \zeta \in \mathbf{C}^+.$$

We say that a subspace M in H^p is a shift-operator-invariant space if

$$S^* M \subset M.$$

It is a celebrating result that M is a shift-operator-invariant space if and only if

$$M = IH^p,$$

where I is an inner function in the context ([13]). This result in the unit disc context is the Beurling Theorem and in the upper-half complex plane context the Beurling-Lax Theorem

([14]). As consequence of these theorems a subspace is a backward-shift-invariant space ($SM \subset M$) if and only if ([15])

$$M = H^p \cap I\overline{H^p}.$$

So, when the inner function is reduced to a Blaschke product, we obtain the right hand side of (1.7) as a particular case.

The results that $\{e_n\}$ and $\{\beta_n\}$ are Schauder bases in their respectively defined backward shift invariant spaces are significant in the H^p and L^p approximation by using these bases. The nature of the problem is non-linear: It deals with a representation of non-linear phase; and the corresponding Christoffel-Darboux's kernel associated with the n-th partial sum is a function of two variables that does not give rise to a convolution operator. The kernel method to prove the same result in the classical Fourier analysis case is not applicable here ([3]). The technical approach of this paper is a blend of an adaptation of a Kolmogorov's method, M. Riesz's theorem on L^p boundedness of the Hilbert transformation and Hilbert-transformation-eigenfunction properties of boundary limits of functions in the Hardy space.

2 TM Systems Are Schauder Bases

We treat the two contexts together. We recall that the system consisting of the orthogonal rational functions in the H^p context has been denoted by $\{\alpha_n\}$. In the unit disc context $\{\alpha_n\} = \{e_n\}$; and in the upper-half-plane context $\{\alpha_n\} = \{\beta_n\}$. We denote the parameter sequence giving rise to the TM system by $\{b_n\}$. In the two contexts it is, respectively, identical with $\{a_n\} \subset \mathbf{D}$ and $\{\lambda_n\} \subset \mathbf{C}^+$. We will be working with the partial sums S_n, defined by

$$S_n(f) = \sum_{k=1}^{n} \langle f, \alpha_n \rangle \alpha_n,$$

where $\langle \cdot, \cdot \rangle$ denotes the paring between H^p and $H^{p'}$, where $1/p + 1/p' = 1, 1 < p, p' < \infty$. The paring in the unit disc context is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} dt;$$

and in the upper-half-complex plane context is

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt.$$

Below, by $\|\cdot\|_p$ we mean either the norm of the Hardy space H^p or that of the boundary Hardy space, depending on the context.

Theorem 2.1 *The linear operators S_n are uniformly bounded in $\overline{\mathrm{span}}^p\{\alpha_n\}$. That is, there exists a constant C such that for all n,*

$$\|S_n f\|_p \le C\|f\|_p, \qquad f \in \overline{\mathrm{span}}^p\{\alpha_n\}. \tag{2.8}$$

Theorem 2.2 *The linear operators S_n are uniformly bounded from $\overline{\mathrm{span}}^p\{\alpha_n\}$ to $\overline{\mathrm{span}}^p\{\alpha_n\}$ if and only if*

$$\lim_{n\to 0} S_n(f) = f \qquad \text{in the } H^p - \text{norm sense}, \quad f \in \overline{\mathrm{span}}^p\{\alpha_n\}. \tag{2.9}$$

Corollary 2.3 *The TM system $\{e_n\}$ is a Schauder basis in $\overline{\mathrm{span}}^p\{e_n\}, 1 < p < \infty$. In other words, for any $f \in \overline{\mathrm{span}}^p\{e_n\}$, we have*

$$f = \sum_{n=1}^{\infty}\langle f, e_n\rangle e_n \qquad \text{in the } H^p(\partial D) - \text{norm sense.}$$

Corollary 2.4 *The TM system $\{\beta_n\}$ is a Schauder basis in $\overline{\mathrm{span}}^p\{\beta_n\}, 1 < p < \infty$. In other words, for any $f \in \overline{\mathrm{span}}^p\{\beta_n\}$, we have*

$$f = \sum_{n=1}^{\infty}\langle f, \beta_n\rangle \beta_n \qquad \text{in the } H^p(\mathbf{R}) - \text{norm sense.}$$

Only Theorem (2.1) and Theorem (2.2) need to be proved. We will first prove Theorem (2.2).

Proof of Theorem (2.2) First we assume that (2.9) holds. In the case the Resonance Theorem implies that S_n is uniformly bounded. Next we assume that S_n is uniformly bounded. We note that due to the formal orthogonal property of the system $\{\alpha_n\}$ under the paring between H^p and $H^{p'}$, the elements α_n all satisfy the condition (2.9), and thus so do all the functions in $\mathrm{span}\{\alpha_n\}$. In the case to show that $\{\alpha_n\}$ is a Schauder basis of $\overline{\mathrm{span}}^p\{\alpha_n\}$ it suffices to show that the functions in H^p satisfying the condition (2.9) form a closed set. Let $f \in \overline{\mathrm{span}}^p\{\alpha_n\}$ be fixed. For any > 0, there exists $f_\epsilon \in \mathrm{span}\{\alpha_n\}$ such that

$$\|f - f_\epsilon\|_p \leq \ .$$

Now, by assumption, S_n are uniformly bounded, therefore,

$$\begin{aligned}\|f - S_n(f)\| &\leq \|f - f_\epsilon\|_p + \|f_\epsilon - S_n(f_\epsilon)\|_p + \|S_n(f - f_\epsilon)\|_p \\ &\leq \ + \|f_\epsilon - S_n(f_\epsilon)\|_p + C \ .\end{aligned}$$

Since $f_\epsilon \in \mathrm{span}\{\alpha_n\}$, for large enough n, we have

$$f_\epsilon = S_n(f_\epsilon), \quad \text{therefore,} \quad \|f_\epsilon - S_n(f_\epsilon)\|_p = 0.$$

For the same large n, being selected according to f and , we have

$$\|f - S_n(f)\|_p \leq (1 + C) \ .$$

The proof is complete.

Proof of Theorem (2.1) Assume that $f \in H^2 \cap \overline{\text{span}}^p\{\alpha_n\}$. Then, by the H^2-theory, we have

$$f = S_n(f) + R_n(f),$$

where $R_n = \sum_{k=n+1}^{\infty} \langle f, \alpha_k \rangle \alpha_k \in H^2$. Let B_n be the finite Blaschke product having $\{\alpha_1, ..., \alpha_n\}$ as all its zeros including the multiples. Then

$$B_n^{-1} f = B_n^{-1} S_n(f) + B_n^{-1} R_n.$$

It is clear that $B_n^{-1} S_n(f) \in H^p(\mathbf{C} \setminus \overline{\mathbf{X}})$ and $B_n^{-1} R_n \in H^p(\mathbf{X})$, where $\mathbf{X}$ is $\mathbf{D}$ or $\mathbf{C}^+$, depending on the context. Note that Hilbert transformation $\mathcal{H}$ has all the functions in $H^2(\mathbf{X})$ as its eigenfunctions with the eigenvalue $-i$; and all the functions in $H^2(\mathbf{C} \setminus \overline{\mathbf{X}})$ as its eigenfunctions with the eigenvalue i. We hence have

$$\mathcal{H} B_n^{-1} f = i B_n^{-1} S_n(f) - i B_n^{-1} R_n.$$

Consequently,

$$B_n \mathcal{H} B_n^{-1} f = i S_n(f) - i R_n.$$

On the other hand,

$$i f = i S_n(f) + i R_n.$$

Therefore,

$$2 i S_n(f) = i f + B_n \mathcal{H} B_n^{-1} f.$$

Note that for any Blaschke product B we have $|B| = 1$, a.e., on the boundary. Due to the uniform boundedness of the multiplication operators by B_n and B_n^{-1}, and the L^p- boundedness of the Hilbert transformation $\mathcal{H}$, we have

$$\|S_n(f)\|_p \leq C \|f\|_p$$

uniformly in n for f in $H^2 \cap \overline{\text{span}}^p\{\alpha_n\}$. By a routine density argument the uniform boundedness of S_n can be extended to the closure of $H^2 \cap \overline{\text{span}}^p\{\alpha_n\}$ in H^p, that is $\overline{\text{span}}^p\{\alpha_n\}$, as desired.

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