Non-stretch mappings for a sharp estimate of the Beurling-Ahlfors operator *

Xingdi Chen [a,b] † and Tao Qian [b]
[a] Department of Mathematics, Huaqiao University, Quanzhou, Fujian 362021, P. R. China
[b] Department of Mathematics, Faculty of Science and Technology, University of Macau, PO Box 3001, P. R. China

Abstract: In this paper we identify certain classes of non-stretch mappings that enjoy a sharp estimate of the Beurling-Ahlfors operator. We first make use of a property of subharmonic functions to prove that the Bañuelos-Wang conjecture and the Iwaniec conjecture are true for a class of mappings that satisfy a quasilinear conjugate Beltrami equation. By utilizing the principal solutions of Beltrami equations, we further explicitly construct some classes of non-stretch mappings for which the Bañuelos-Wang conjecture and the Iwaniec conjecture are true.

Keywords: Beurling-Ahlfors operator; Cauchy operator; Harmonic mapping; Beltrami equation; Principal solution.
2000 Mathematics Subject Classification. Primary 30C62; Secondly 47G10, 42B20

1 Introduction

The *Beurling-Ahlfors operator* $\mathbf{T}$ is defined on $L^p(\mathbb{C})$, $1 < p < \infty$, by

$$\mathbf{T}f(z) = -\frac{1}{\ }\mathrm{pv}\iint_{\mathbb{C}} \frac{f(\ )}{(z-\ )^2}dm(\ ); \qquad (1.1)$$

where pv means the Cauchy principal value and m is the Lebesgue measure in the plane $\mathbb{C}$. The Beurling-Ahlfors operator arises naturally in the study of the

* **Foundation items** Supported by NNSF of China (11101165), NCETFJ Fund (2012FJ-NCET-ZR05), Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (ZQN-YX110) and MYRG115(Y1-L4)-FST13-QT.
† Corresponding author. Email-address: chxtt@hqu.edu.cn

solutions of Beltrami equations [3, 5]. This operator and its multidimensional analogues are fundamental tools in several areas including quasiconformal mappings, partial differential equations, calculus of variations and differential geometry (see [3, 4, 5, 8, 18, 25, 30] and the references therein for more details).

For a function $f = u + iv : \mathbb{C} \to \mathbb{C}$, we denote its formal partial derivatives by

$$\bar{\partial} f = f_{\bar z} = \frac{1}{2}(f_x + i f_y) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)),$$
$$\partial f = f_z = \frac{1}{2}(f_x - i f_y) = \frac{1}{2}(u_x + v_y + i(v_x - u_y)),$$

and write

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Let $\dot{W}^{1,p}(\mathbb{C},\mathbb{C})$, $1 < p < \infty$, be the homogenous Sobolev space of complex-valued locally integrable functions in the plane whose distributional first derivatives are in $L^p(\mathbb{C})$. A function $f : \mathbb{C} \to \mathbb{C}$ is called *radial* if $f(re^{i}) = g(r)$, while, f is said to be a *stretch mapping* if it is of the form $f(re^{i}) = g($)z=f i W, ;p CC S17 11.9552 Tf 7.113 -4.774(1) CCifCCC

The validity of the Šverák conjecture implies that of the Bañuelos-Wang conjecture (see Section 1 in [7] for a proof). By the ***Burkholder inequality*** (p16-17 in [13])

$$p(1-\frac{1}{p^*})^{p-1}((p^*-1)|\bar{\partial} f|-|\partial f|)(|\bar{\partial} f|+|\partial f|)^{p-1} \leq (p^*-1)^p|\bar{\partial} f|^p-|\partial f|^p; \quad (1.5)$$

the Bañuelos-Wang conjecture in turn implies the Iwaniec conjecture.

In 1952, Morrey [28] conjectured that the rank-one convexity of a functional $\mathbf{F} : M(m; n) \to \mathbb{R}$ does not imply its quasiconvexity when both m and n are at least 2, where $M(m; n)$ denotes the set of all $m \times n$ matrices with real entries. Due to the rank-one convexity of the Burkholder functional and the Šverák functional, the above three conjectures are also closely connected with the Morrey conjecture. One can see Section 5 in [7] or [32] for a precise statement of these relations.

Bañuelos and Wang [11] used martingale inequalities [13] to show that $||\mathbf{T}||_{L^p(\mathbb{C})} \leq 4(p^*-1)$. Utilizing an analytic approach with Bellman functions, Nazarov and Volberg [29] improved it and got $2(p^*-1)$. So far, the best result is $||\mathbf{T}||_{L^p(\mathbb{C})} \leq 1.575(p^*-1)$, obtained by Bañuelos and Janakiraman [9] by probabilistic techniques of Burkholder [13, 14]. One can refer to [12, 21] for its asymptotical estimates and see [19, 20] for the L^p-norm estimates of the powers $\mathbf{T}^n$.

On one hand, there have been efforts to decrease the constant C in the inequality

$$||\mathbf{T}f||_{L^p(\mathbb{C})} \leq C(p^*-1)||f||_{L^p(\mathbb{C})} \quad (1.6)$$

for all functions $f \in L^p(\mathbb{C})$, while, on the other hand, there were results establishing this inequality with $C = 1$ but just for particular subclasses of $L^p(\mathbb{C})$.

Baernstein and Montgomery-Smith [7] showed that the Bañuelos-Wang conjecture holds for every stretch mapping $f \in S \cap \dot{W}^{1,p}(\mathbb{C};\mathbb{C})$ and consequently the Iwaniec conjecture is valid for this class of mappings. Recently, Volberg [32] extended the above result to complex radial functions.

Theorem A. *If a complex valued function f has an expression*

$$f(z) = f(|z|), \quad f \in C_0^\infty(\mathbb{C}),$$

then it follows

$$||\mathbf{T}f||_{L^p(\mathbb{C})} \leq (p^*-1)||f||_{L^p(\mathbb{C})}. \quad (1.7)$$

Let $\mathbb{H}$ be a separable Hilbert space over $\mathbb{R}$ with norm $|\cdot|$ and scalar product $< \cdot, \cdot >$, and $F : \mathbb{C} \to \mathbb{H}$ belong to $L^p(\mathbb{C})$. Bañuelos and Osękowski [10] used martingale inequalities to show that the inequality (1.7) holds for all radial functions F and the constant p^*-1 is the best possible for $1 < p \leq 2$.

Let Ω be a simply-connected domain of $\mathbb{C}$. Recall that a ***harmonic mapping*** f defined on Ω is a solution of the conjugate Beltrami equation

$$\overline{f_{\bar{z}}} = a f_z \quad (1.8)$$

in $W^{1,2}_{loc}(\Omega)$, where a is analytic and $|a| < 1$ on Ω. We refer to [17, 22, 23] for the study of harmonic mappings. In [7], Baernstein and Montgomery-Smith proved the following

Theorem B. *If* $f \in \dot{W}^{1,p}(\mathbb{C};\mathbb{C})$, $1 < p < \infty$, *is harmonic on* $\mathbb{C} \cup \infty \backslash \{|z| = 1\}$, is called the ***Neumann series***. When satisfies $\|\ \|_\infty \le k < 1$ and has a compact support, the Neumann series converges in $L^p(\mathbb{C})$ norm, where k is a constant (see p163 in [5]). If is degenerative, *i.e.*, $\|\ \|_\infty$=1, the convergence of the Neumann series is not easy to be determined. For some particular classes of degenerative Beltrami coefficients , the convergence of their Neumann series can be determined if there exist explicit representations of $\mathbf{C}$ and $\mathbf{T}$ (see Lemma 3.1).

If the conjugate of a Beltrami coefficient is analytic, then we call it a ***co-analytic Beltrami coe cient***. Let I be the identical mapping in this text. We show that if $f + I$ is a principal solution with a co-analytic Beltrami coefficient, then the Bañuelos-Wang conjecture and the Iwaniec conjecture are true for f (see Theorem 3.1).

Moreover, using the Parseval formula we give two classes of principal solutions $f + I$ with degenerative Beltrami coefficients that enable the corresponding mappings f validating the Bañuelos-Wang conjecture and the Iwaniec conjecture for $p = 2$ and $p = 4$ (see Example 3.2 and Theorem 3.2). We note that these mappings are not stretch or complex radial.

This rest of this paper is organized as follows. In Section 2, using the fact that the integral means of a subharmonic function are non-decreasing, we obtain the proof of Theorem 1.1. In Section 3, we use principle solutions to construct several classes of non-stretch mappings that validate the Bañuelos-Wang conjecture and the Iwaniec conjecture.

2 Proof of Theorem 1.1

Proof. By the assumption that $g \in W^{1;2}_{loc}(\mathbb{D})$ and $|a| < 1$, we have that, as a solution of (1.9), g is a locally quasiregular mapping of $\mathbb{D}$. Consequently, it is open and sense preserving. Denote by $\mathbb{Z}(g)$ the zero set of g. For any point $z_0 \in \mathbb{D}\backslash\mathbb{Z}(g)$, there exists a $r > 0$ such that $\log g$ is harmonic on $\mathbb{D}(z_0; r) = \{z||z - z_0| < r\}$ and thus $g \in C^\infty(\mathbb{D}(z_0; r))$. Hence, by (1.10) we have g is $\frac{1}{|g|^2}$-harmonic on $\mathbb{D}(z_0; r)$, that is, g satisfies

$$gg_{zz} = g_z g_z; \quad z \in \mathbb{D}(z_0; r): \tag{2.1}$$

Differentiating both sides of (2.1) in z, we obtain

$$g_{zzz} = \frac{g_{zz} g_z}{g}; \quad z \in \mathbb{D}(z_0; r):$$

The assumption of the locally univalence of g implies that $\log g$ is locally univalent on $\mathbb{D}(z_0; r)$. By the Lewy theorem [27], the harmonicity of $\log g$ on $\mathbb{D}(z_0; r)$ implies that the Jacobian $J_{\log g} > 0$ on $\mathbb{D}(z_0; r)$ and consequently $|g_z| > 0$ on $\mathbb{D}(z_0; r)$. Multiplying g_z to both sides of the above equality, we have

$$g_z g_{zzz} = g_{zz} g_{zz}; \quad z \in \mathbb{D}(z_0; r):$$

Direct computation shows that

$$\Delta \log |g_z| = 0 \tag{2.2}$$

holds for all $z \in \mathbb{D}\backslash\mathbb{Z}(g)$. This implies that $\log|g_z|$ is subharmonic on $\mathbb{D}$. The relation (1.9) and the subharmonicity of $\log|g_z|$ and $\log|a|$ show that $\log|g_z|$ is also subharmonic on $\mathbb{D}$. Hence, the logarithms of both $|g_z|(|g_z|+|g_z|)^{p-1}$ and $|g_z|(|g_z|+|g_z|)^{p-1}$ are subharmonic on $\mathbb{D}$. Thus, the functions themselves are subharmonic on $\mathbb{D}$.

Let $f = g \circ {}'$ and $= \frac{1}{z}$. For any $z \in \mathbb{D}^c$, it follows that

$$f_z = (g \circ {}')_z = (g(\frac{1}{\bar{z}}))_z = g\;(\;)\;_z + g\;(\;)\;\bar{}_z = -\bar{}^2 g\;(\;); \tag{2.3}$$

and

$$f_z = (g \circ {}')_z = (g(\frac{1}{\bar{z}}))_z = g\;(\;)\;_z + g\;(\;)\;\bar{}_z = -\;{}^2 g\;(\;): \tag{2.4}$$

For $z \in \mathbb{D}$, we have

$$f_z = (g \circ {}')_z = g_z; \quad f_z = (g \circ {}')_z = g_z: \tag{2.5}$$

By the definition of $\mathbf{B}_p(Df)$ and the assumption that $f \in \dot{W}^{1;p}(\mathbb{C};\mathbb{C})$, we get from (2.3), (2.4) and (2.5) that

$$\begin{aligned}
\iint \mathbf{B}_p(Df)dm &= \iint_{\mathbb{D}} \mathbf{B}_p(Df)dm + \iint_{\mathbb{D}^c} \mathbf{B}_p(Df)dm \\
&= \iint_{\mathbb{D}} [(p^*-1)|g\;| - |g\;|](|g\;|+|g\;|)^{p-1}dm(\;) \\
&+ \iint_{\mathbb{D}} [(p^*-1)|g\;| - |g\;|](|g\;|+|g\;|)^{p-1}|\;|^{2(p-2)}dm(\;) \\
&= \iint_{\mathbb{D}} [(p^*-1) - |\;|^{2(p-2)}]|g\;|(|g\;|+|g\;|)^{p-1}rdrd \\
&+ \iint_{\mathbb{D}} [(p^*-1)|\;|^{2(p-2)} - 1]|g\;|(|g\;|+|g\;|)^{p-1}rdrd \;= I + II;
\end{aligned}$$

where $= re^{i}$. It is clear that $I = II = 0$ when $p = 2$. If $2 < p < \infty$, then $I > 0$. Now we can also show that $II > 0$.

Write

$$I_1(r) = \frac{1}{2}\int_0^2 |g\;|(|g\;|+|g\;|)^{p-1}d\;; \quad I_2(r) = \frac{1}{2}\int_0^2 |g\;|(|g\;|+|g\;|)^{p-1}d\;: \tag{2.6}$$

Then II can be written as

$$II = 2\int_0^1 [(p-1)r^{2p-3} - r]I_1(r)dr:$$

Integration by parts gives

$$II = 2\int_0^1 (\frac{r^2}{2} - \frac{r^{2p-2}}{2})dI_1(r): \tag{2.7}$$

When $2 < p < \infty$, the inequality $\frac{r^2}{2} - \frac{r^{2p-2}}{2} > 0$ holds for $0 < r < 1$. The subharmonic property of the integrand of $I_1(r)$ implies that $I_1(r)$ is non-decreasing for $0 < r < 1$, that is, $dI_1(r) \geq 0\ a.e.$. Hence, $II > 0$.

When $1 < p < 2$, $II > 0$ is obvious and the inequality $I > 0$ can be deduced from the non-decreasing property of $I_2(r)$ on $(0, 1)$ and the technique that we use in the case $2 < p < \infty$. Thus, for $1 < p < \infty$, we have

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm \geq 0.$$

So, the Bañuelos-Wang conjecture is true for a mapping $f = g \circ \varphi \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, when g satisfies the partial differential equation (1.9). As a consequence, the I-for $f \in L^p(\mathbb{C})$. Write

$$\mathbf{Q}^n f = \underbrace{\mathbf{Q} \circ \cdots \circ \mathbf{Q}}_{n}(f); \quad n \in \mathbb{N}^+.$$

By induction, $\mathbf{Q}^n(\quad)$ is well defined for all $n \in \mathbb{N}^+$. If the series $\sum_{n=1}^{\infty} \mathbf{Q}^n(\quad)$ converges and its sum h belongs to $L^p(\mathbb{C})$, $p \geq 2$, then

$$f = z + \mathbf{C}(\quad + h) \tag{3.3}$$

is a principal solution of the Beltrami equation

$$f_{\bar{z}} = \quad f_z.$$

Moreover, $f_z - 1 \in L^p(\mathbb{C})$, $p \geq 2$, and

$$f_z = 1 + \mathbf{T}(\quad + h); \quad f_{\bar{z}} = \quad + h.$$

Lemma 3.1. *Let* $\quad = \bar{z}^n z^m$*, where* n *and* m *are integers. Then the following relations hold. If* $n \geq m$*, then*

$$\mathbf{C}(\quad_{\mathbb{D}})(z) = z^m \frac{'(\bar{z})^{n+1}}{n+1} \tag{3.4}$$

and

$$\mathbf{T}(\quad_{\mathbb{D}})(z) = \begin{cases} \frac{m}{n+1} z^{m-1}\bar{z}^{n+1}; & m \neq 0; z \in \mathbb{D}; \\ 0; & m = 0; z \in \mathbb{D}; \\ -\frac{n-m+1}{(n+1)z^{n-m+2}}; & z \in \mathbb{D}^c. \end{cases} \tag{3.5}$$

If $n = m - 1$*, then*

$$\mathbf{C}(\quad_{\mathbb{D}})(z) = -\frac{1 - |z|^{2n+2}}{n+1} \quad_{\mathbb{D}} \tag{3.6}$$

and

$$\mathbf{T}(\quad_{\mathbb{D}})(z) = z^n \bar{z}^{n+1} \quad_{\mathbb{D}}. \tag{3.7}$$

If $n \leq m - 2$*, then*

$$\mathbf{C}(\quad_{\mathbb{D}})(z) = -\frac{z^{m-(n+1)}}{n+1}(1 - |z|^{2n+2}) \quad_{\mathbb{D}} \tag{3.8}$$

and

$$\mathbf{T}(\quad_{\mathbb{D}})(z) = \left(-\frac{m-(n+1)}{n+1} z^{m-(n+2)} + \frac{m}{n+1} z^{m-1}\bar{z}^{n+1}\right) \quad_{\mathbb{D}}. \tag{3.9}$$

Proof. Let $\quad = re^{i}$. By the definition of the Cauchy operator, we have

$$\mathbf{C}(\quad_{\mathbb{D}})(z) = -\frac{1}{}\iint_{\mathbb{C}} \frac{\bar{\quad}^n \quad^m \quad_{\mathbb{D}}}{\quad - z} dm(\quad) = -2 \int_0^1 r^{2n+1} I_z(r) dr; \tag{3.10}$$

where

$$I_z(r) = \frac{1}{2\pi i}\oint_{|\zeta|=r} \frac{1}{\zeta^{n-m+1}(\zeta - z)}d\zeta. \tag{3.11}$$

When $n \geq m$, we obtain

$$I_z(r) = -\frac{1}{z^{n-m+1}}\chi_{\mathbb{D}}. \tag{3.12}$$

Thus, it follows from (3.12) that

$$\mathbf{C}(\varphi\chi_{\mathbb{D}})(z) = -2\int_0^1 r^{2n+1} I_z(r)dr = \frac{1}{(n+1)z^{n-m+1}}, \quad z \in \mathbb{D}^c$$

and

$$\mathbf{C}(\varphi\chi_{\mathbb{D}})(z) = -2\int_0^{|z|} r^{2n+1} I_z(r)dr + \int_{|z|}^1 r^{2n+1} I_z(r)dr = \frac{1}{(n+1)} z^m \bar{z}^{n+1}, \quad z \in \overline{\mathbb{D}}.$$

By the first equality of (3.2) of Lemma A, one can get (3.5).

The proofs of the cases $n = m - 1$ and $n \leq m - 2$ can be obtained by the method used in the case $n \geq m$, we omit for simplicity. □

Example 3.1. Let $\varphi = z$. Then a principal solution of the Beltrami equation

$$f_{\bar{z}} = \varphi\chi_{\mathbb{D}} f_z$$

is given by

$$f(z) = ze^{\varphi(z)} - 1 \tag{3.13}$$

Proof. Choose $m = 1, n = 0$ in Lemma 3.1. Then by the relation (3.7), we have

$$\mathbf{Q}(\varphi\chi_{\mathbb{D}}) = z\bar{z}\chi_{\mathbb{D}}.$$

The relation (3.5) gives

$$\mathbf{Q}^2(\varphi\chi_{\mathbb{D}}) = \frac{1}{2}z\bar{z}^2\chi_{\mathbb{D}}.$$

Hence, it follows from induction that

$$\mathbf{Q}^n(\varphi\chi_{\mathbb{D}}) = \frac{1}{n!}z\bar{z}^n\chi_{\mathbb{D}}, \quad n \in \mathbb{N}^+.$$

Set $\mathbf{Q}^0(\varphi\chi_{\mathbb{D}}) = \varphi\chi_{\mathbb{D}}$. By the convergence of the series $\sum_{n=0}^{\infty}\mathbf{Q}^n(\varphi\chi_{\mathbb{D}})$ and the fact that its sum belongs to $L^p(\mathbb{C})$, $p \geq 2$, we have that $f = z + \mathbf{C}(\sum_{n=0}^{\infty}\mathbf{Q}^n(\varphi\chi_{\mathbb{D}}))$ is a principal solution of the Beltrami equation $f_{\bar{z}} = z\chi_{\mathbb{D}} f_z$. Moreover, for $z \in \mathbb{D}$,

$$\begin{aligned} f(z) &= z + \mathbf{C}(\sum_{n=0}^{\infty}\mathbf{Q}^n(\varphi\chi_{\mathbb{D}})) \\ &= z - (1 - |z|^2) + z(\frac{1}{2}\bar{z}^2 + \frac{1}{3\cdot 2!}\bar{z}^3 + \cdots + \frac{1}{(n+1)\cdot n!}\bar{z}^{n+1} + \cdots) \\ &= ze^{\bar{z}} - 1. \end{aligned}$$

Similarly, for $z \in \mathbb{D}^c$, we have $f(z) = ze^{\frac{1}{z}} - 1$. □

Next, we will use principal solutions to construct several classes of mappings validating the Bañuelos-Wang conjecture and the Iwaniec conjecture.

Theorem 3.1. *Let I be the identical mapping and is co-analytic on $\mathbb{C}$. If $f + I$ is a principal solution with the Beltrami coe cient $_{\mathbb{D}}$, then*

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm \geq 0; \tag{3.14}$$

and the equality holds when $p = 2$.

Proof. The assumption on implies that can be represented by a power series $\sum_{n=0} \overline{a_n}\bar{z}^n$. Owing to (3.5), we have that $_{\mathbb{D}}\mathbf{T}(\bar{z}^n \ _{\mathbb{D}}) = 0$ for all $n \in \mathbb{N}^+$. Now the linearity of the Beurling-Ahlfors operator implies

$$\mathbf{Q} \quad _{\mathbb{D}}(z) = 0:$$

So,

$$\mathbf{Q}^n(\quad _{\mathbb{D}}) = 0; \quad n \in \mathbb{N}^+:$$

By the linearity of the Cauchy operator, we get

$$f + I = z + \mathbf{C}(\sum_{n=0} \mathbf{Q}^n(\quad _{\mathbb{D}})) = z + \sum_{n=0} \mathbf{C}(\overline{a_n z^n} \ _{\mathbb{D}}):$$

According to (3.4), we have

$$f(z) = \sum_{n=0} \mathbf{C}(\overline{a_n z^n} \ _{\mathbb{D}}) = \sum_{n=0} \overline{a_n} \frac{'(\bar{z})^{n+1}}{n+1}: \tag{3.15}$$

Now we prove that f validates the Bañuelos-Wang conjecture.

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm = \iint_{\mathbb{D}} \mathbf{B}_p(Df)dm + \iint_{\mathbb{D}^c} \mathbf{B}_p(Df)dm = I + II:$$

By (3.15), we have

$$I = \iint_{\mathbb{D}} (p-1)| \sum_{n=0} a_n z^n|^p dxdy;$$

and

$$II = \iint_{\mathbb{D}^c} | \sum_{n=0} \overline{a_n} \frac{1}{z^{n+2}}|^p dxdy = \iint_{\mathbb{D}} | \sum_{n=0} a_n z^{n+2}|^p |z|^{-4} dm(z):$$

Let $z = re^{i}$. Then,

$$\iint_{\mathbb{C}} \mathbf{B}_p(Df)dm = \iint_{\mathbb{D}} | \quad |^p((p-1) - r^{2(p-2)}$$

Generally, it is difficult to explicitly represent a principal solution for a given Beltrami coefficient. For some special classes of Beltrami coefficients, we can obtain their explicit principal solutions and use them to construct non-stretch examples validating the Bañuelos-Wang conjecture and the Iwaniec conjecture.

Example 3.2. Let $g(z) = f(z) - z + 1$, where $f(z)$ is given by Example 3.1. Then

$$\iint_{\mathbb{C}} \mathbf{B}_2(Dg)dm = 0; \qquad \iint_{\mathbb{C}} \mathbf{B}_4(Dg)dm > 0:$$

Proof. By the equation (3.13), we get

$$g_z = \begin{cases} e^z - 1; & |z| < 1; \\ e^{\frac{1}{z}} - \frac{1}{z}e^{\frac{1}{z}} - 1; & |z| > 1; \end{cases} \qquad g_z = \begin{cases} ze^z; & |z| < 1; \\ 0; & |z| > 1: \end{cases} \tag{3.16}$$

It follows from the Parseval formula that

$$\iint_{\mathbb{D}} (|ze^z|^2 - |e^z - 1|^2)dm(z) = \sum_{n=2} \frac{n-1}{(n!)^2} \tag{3.17}$$

and

$$\iint_{\mathbb{D}} \frac{|e^z - ze^z - 1|^2}{|z|^4} dm(z) = \sum_{n=2} \frac{n-1}{(n!)^2}: \tag{3.18}$$

By the above two equations we have

$$\iint_{\mathbb{C}} \mathbf{B}_2(Dg)dm = \iint_{\mathbb{D}} (|ze^z|^2 - |e^z - 1|^2 - \frac{|e^z - ze^z - 1|^2}{|z|^4})dm(z) = 0:$$

From the power series $\sum_{n=0} \frac{z^n}{n!}$ of e^z, it follows that

$$z^2e^{2z} = \sum_{n=2} \frac{2^{n-2}}{(n-2)!}z^n; \quad (e^z - 1)^2 = \sum_{n=2} \frac{2^n - 2}{n!}z^n; \tag{3.19}$$

$$(\frac{e^z - ze^z - 1}{z})^2 = \sum_{n=2} \frac{2^n(n-2) + 2}{n+2}\frac{z^n}{n!}: \tag{3.20}$$

Next we prove the second assertion of Example 3.2. By direct calculations, we have

$$\begin{aligned} \iint_{\mathbb{C}} \mathbf{B}_4(Dg)dm &= \iint_{\mathbb{C}} (3|g_z|^4 - |g_z|^4 + 6|g_z|^2|g_z|^2 + 8|g_z||g_z|^3)dm \\ &\geq \iint_{\mathbb{C}} (3|g_z|^4 - |g_z|^4)dm(z) = III - IV; \end{aligned}$$

where

$$III = \iint_{\mathbb{D}} [3|z^2e^{2z}|^2 - |(e^z - 1)^2|^2]dm(z); \quad IV = \iint_{\mathbb{D}} \frac{|(e^z - ze^z - 1)^2|^2}{|z|^4}dm(z):$$

Using the Parseval formula, we obtain from (3.19) and (3.20) that

$$\begin{aligned} III - IV &= \sum_{n=2}^{\infty} \frac{1}{[(n-2)!]^2(n+1)}\Big\{\frac{3\cdot 2^{2n}}{16} - \frac{(2^n-2)^2(n+2)^2+[2^n(n-2)+2]^2}{[(n-1)n(n+2)]^2}\Big\} \\ &\geq \Big\{\frac{31}{16} + \sum_{n=3}^{\infty} \frac{11}{144}\frac{4^n}{[(n-2)!]^2(n+1)}\Big\} > 0. \end{aligned}$$

The proof of Example 3.2 is now complete. □

Moreover, we can get a more general result as follows

Theorem 3.2. *Let I be the identical mapping and $\varphi = \bar{z}^n z$ on $\mathbb{C}$, where $n \geq 1$. If $f + I$ is a principal solution of the Beltrami equation with the Beltrami coefficient $\varphi\chi_{\mathbb{D}}$, then*

$$\iint_{\mathbb{C}} \mathbf{B}_4(Df)dm > 0.$$

Proof. By induction, we get from the equality (3.5) at Lemma 3.1 that

$$\mathbf{Q}^k = \begin{cases} \frac{1}{k!}\frac{1}{(n+1)^k}\bar{z}^{k(n+1)}, & |z| \leq 1, \\ -\frac{kn+k+1}{k!(n+1)^k}\frac{1}{z^{kn+k+2}}, & |z| > 1, \end{cases} \tag{3.21}$$

where $k \geq 1$. Hence, by the equality (3.4) of Lemma 3.1 we have

$$\mathbf{C}(\mathbf{Q}^k(\varphi\chi_{\mathbb{D}})) = \begin{cases} \frac{1}{(k+1)!}\frac{1}{(n+1)^{k+1}}\bar{z}^{k(n+1)}z, & |z| \leq 1, \\ \frac{1}{(k+1)!}\frac{1}{(n+1)^{k+1}}\frac{1}{z^{k(n+1)+k}}, & |z| > 1. \end{cases}$$

Then the representation (3.3) gives

$$f(z) = ze^{\frac{\varphi(\bar{z})^{n+1}}{n+1}} - z.$$

Moreover, it follows

$$f_z = \begin{cases} e^{\frac{\bar{z}^{n+1}}{n+1}} - 1, & |z| \leq 1, \\ e^{\frac{1}{(n+1)z^{n+1}}} - \frac{1}{z^{n+1}}e^{\frac{1}{(n+1)z^{n+1}}} - 1, & |z| > 1, \end{cases}$$

and

$$f_{\bar{z}} = z\bar{z}^n e^{\frac{\bar{z}^{n+1}}{n+1}}\chi_{\mathbb{D}}.$$

Using change of variable, we have

$$\begin{aligned} \iint_{\mathbb{C}} \mathbf{B}_4(Df)dm &= \iint_{\mathbb{C}} (3|f_{\bar{z}}|^4 - |f_z|^4 + 6|f_z|^2|f_{\bar{z}}|^2 + 8|f_z||f_{\bar{z}}|^3)dm \\ &\geq \iint_{\mathbb{C}} (3|f_{\bar{z}}|^4 - |f_z|^4)dm = V - VI, \end{aligned}$$

where

$$V = \iint_{\mathbb{D}} [3|z^{2(n+1)}e^{2\frac{z^{n+1}}{n+1}}|^2 - |(e^{\frac{z^{n+1}}{n+1}} - 1)^2|^2]dm(z),$$

and

$$VI = \iint_{\mathbb{D}} \frac{|(e^{\frac{z^{n+1}}{n+1}} - z^{n+1}e^{\frac{z^{n+1}}{n+1}} - 1)^2|^2}{|z|^4} dm(z).$$

From the power series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, it follows

$$(e^{\frac{z^{n+1}}{n+1}} - z^{n+1}e^{\frac{z^{n+1}}{n+1}} - 1)^2$$
$$= \sum_{k=2}^{\infty} \frac{1}{(k-2)!}(2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}((n+1) - \frac{1}{k})(\frac{z^{n+1}}{n+1})^k.$$

Utilizing the Parseval formula, we obtain, from (3.19) and the above relation, that

$$V - VI = 2\sum_{k=2}^{\infty} \frac{1}{((k-2)!(n+1)^k)^2}\{(3 * 2^{2(k-2)}(n+1)^4 - \frac{(2^k-2)^2}{k^2(k-1)^2})$$
$$\frac{1}{2k(n+1)+2} - (2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}((n+1) - \frac{1}{k}))^2 \frac{1}{2k(n+1)-2}\}.$$

The assumptions that $n \geq 1$ and $k \geq 2$ imply that

$$2^{k-2}(n+1)^2 - \frac{2^k-2}{k-1}(n+1-\frac{1}{k}) > (\frac{2^k}{2} - \frac{2^k-2}{k-1})(n+1) \geq 0$$

and

$$2^{2(k-2)}(n+1)^4 - \frac{(2^k-2)^2}{k^2(k-1)^2} > 2^{2k} - \frac{2^{2k}}{4} = \frac{3}{4}2^{2k} > 0.$$

Thus, we have

$$V - VI > 2\sum_{k=2}^{\infty} \frac{1}{((k-2)!(n+1)^k)^2}\{\frac{3}{4}\frac{2^{2k}}{2k(n+1)+2}$$
$$+ 2^{2(k-2)}(n+1)^4 \frac{k(n+1)-3}{2(k(n+1))^2-2)}\} > 0.$$

Therefore, Theorem 3.2 follows. □

Acknowledgements. The authors are deeply indebted to the referees for their very careful work and many helpful suggestions which improved the quality of this paper.

References

[1] Z. Abdulhadi, M. Aydoğan, Integral means and arclength inequalities for typically real logharmonic mappings, *Appl. Math. Lett.* 25 (1) (2012) 27-32.

[2] Z. Abdulhadi, D. Bshouty, Univalent functions in $H \cdot \bar{H}(D)$, *Trans. Amer. Math. Soc.* 305 (2) (1988) 841-849.

[3] L.V. Ahlfors, *Lectures on quasiconformal mappings*, Univ. Lecture Ser. Vol. 38, Amer. Math. Soc., 2006.

[4] K. Astala, Area distortion of quasiconformal mappings, *Acta Math.* 173 (1994) 37-60.

[5] K. Astala, T. Iwaniec, G. Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, 2009.

[6] K. Astala, T. Iwaniec, I. Prause, E. Saksman, Burkholder integrals, Morrey's problem and quasiconformal mappings, *J. Amer. Math. Soc.*, 25 (2) (2012) 507-531.

[7] A. Baernstein II, S.J. Montgomery-Smith, Some conjectures about integral means of ∂f and $\bar{\partial} f$, *Complex analysis and differential equations*, pages 92-109, 1997.

[8] R. Bañuelos, The foundational inequalites of D. L. Burkholder and some of their ramifications, *Illinois J. Math.* 54 (2010) 789-868.

[9] R. Bañuelos, P. Janakiraman, L_p-bounds for the Beurling-Ahlfors transform, *Trans. Amer. Math. Soc.* 360 (7) (2008) 3603-3612.

[10] R. Bañuelos, A. Osękowski, Sharp inequalities for the Beurling-Ahlfors transform on radial functions, *Duke Math. J.* 162 (2) (2013) 417-434.

[11] R. Bañuelos, G. Wang, Sharp inequalities for Martingales with applications to the Beurling-Ahlfors and Riesz transforms, *Duke Math. J.* 80 (1995) 575-600.

[12] A. Borichev, P. Janakiraman, A. Volberg, Subordination by conformal Martingales in L_p and zeros of Laguerre polynomials, *Duke Math. J.* 162 (5) (2013), 889-924.

[13] D. Burkholder, Boundary value problems and sharp estimates for the Martingale transforms, *Ann. of Prob.* 14 (1984) 647-702.

[14] D. Burkholder, Explorations of martingale theory and its applications, *Lecture Notes in Math.* 1464 (1991) 1-66.

[15] X.D. Chen, Hyperbolically bi-Lipschitz continuity for $1/|w|^2$-harmonic quasiconformal mappings, *Int. J. Math. Math. Sci.*, Art.ID569481(2012) 1-13.

[16] X.D. Chen, A.N. Fang. Harmonicity of the inverse of a harmonic diffeomorphism, *J. Math. Anal. Appl.* 389 (1) (2012) 647-655.

[17] J.G. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. AI Marh.* 9 (1984) 3-25.

[18] S. Donaldson, D. Sullivan, Quasiconformal 4-manifolds, *Acta Math.* 163 (1989) 181-252.

[19] O. Dragicevic, Some remarks on the L^p estimates for powers of the Ahlfors-Beurling operator, *Arch. Math.* 96 (2011) 463-471.

[20] O. Dragičević, S. Petermichl, A. Volberg, A rotation method which gives linear L_p estimates for powers of the Ahlfors-Beurling operator, *J. Math.Pures Appl.* 86 (2006) 492-509.

[21] O. Dragičević, A. Volberg, Bellman functions, Littlewood-Paley estimates and asymptotics for the Ahlfors-Beurling operator in $L_p(C)$, *Indiana Univ. Math. J.* 54 (4) (2005) 971-995.

[22] P. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge, Mass, USA, 2004.

[23] J. Eells, L. Lemaire, A report on harmonic maps, *B. Lond. Math. Soc.* 10 (1) (1978) 1-68.

[24] T. Iwaniec, Extremal inequalities in Sobolev spaces and quasiconformal mappings, *Z. Anal. Anwendungen*, 1 (1982) 1-16.

[25] T. Iwaniec, Nonlinear Cauchy-Riemann operators in R^n, *Trans. Amer.Math. Soc.* 354 (2002) 1961-1995.

[26] O. Lehto, Remarks on the integrability of the derivatives of quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. AI Math.* 371 (1965) 8pp.

[27] H. Lewy, On the non-vanishing of the Jacobian in certain one to one mapings, *Bull. Amer. Math. Soc.* 42 (1936) 689-692.

[28] C.B. Morrey, Quasi-convexity and the lower semicontinuity of multiple integrals, *Paci c J. Math.* 2 (1952) 25-53.

[29] F. Nazarov, A. Volberg, Heat extension of the Beurling operator and estimates for its norm, *(Russian) Algebra i Analiz*, 15 (4) (2003) 142-158.

[30] S. Petermichl, A. Volberg, Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular, *Duke Math. J.* 112 (2) (2002) 281-305.

[31] V. Šverák, Rank-one convexity does not imply quasiconvexity, *Proc. Roy. Soc. Edinburgh*, 120 (A) (1992) 185-189.

[32] A. Volberg, Ahlfors-Beurling operator on radial functions, *arXiv:1203.2291*.