Sufficient conditions that the shift-invariant system is a frame for $L^2(\mathbb{R}^n)$[1]

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Abstract. The main contribution of this paper is twofold. First, some new su±cient conditions, under which the shift-invariant system is a frame for $L^2(\mathbb{R}^n)$, were established. Second, those su±cient conditions are used to obtain some new results about Gabor frames.

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1. Introduction

The theory of frames, or stable redundant non-orthogonal expansions in Hilbert spaces, introduced by Du±n and Schae®er ([1]), plays an important role in wavelet theory ([2-3]) as well as in Gabor analysis ([4-6]) for functions in $L^2(\mathbb{R}^n)$. Traditionally, frames were used in signal processing because of their resilience to additive, resilience to quantization and because they give greater °exibility to capture important signal characteristic([7-8]). Today, frame theory has a myriad of applications in pure mathematics, applied mathematics and even wireless communication ([9-10]).

In [11], by the study of shift-invariant subspaces of $L^2(\mathbb{R}^n)$, the author presented a general result to characterize tight frame generators for the shift-invariant subspaces. Some applications of this general result are then obtained, among which are the characterization of tight wavelet frames and tight Gabor frames.

In [12], some necessary and su±cient conditions ensuring that the shift-invariant system is a frame for $L^2(\mathbb{R}^n)$ were established, and then some known results about wavelet frames and Gabor frames were obtained by applying these conditions.

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In present paper, we will give some new su±cient conditions under which the shift-invariant system is a frame for $L^2(\mathbb{R}^n)$. As some applications, the results are used to obtain some new conclusions about Gabor frames.

This paper is organized as follows. Section 2 includes some notations, de¯nitions and auxiliary results. In order to prove the main results, some lemmas are given in Section 3. Section 4 is devoted to the discussion of su±cient conditions for the shift-invariant system to be a frame. Section 5 contains applications of main results to Gabor frames.

2. Preliminaries

Some notations, de¯nitions and auxiliary results are listed in this section.

Firstly, several de¯nitions and a fact about the frames are given. Additional information on the subject can be found in [3-9].

Definition 2.1 *Suppose that H is a separable Hilbert space and a countable family $\{x_i \mid i \in I\} \subset H$. If there exist constants $0 < A \le B < \infty$ such that $\forall x \in H$,*

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B\|x\|^2, \tag{1}$$

then $\{x_i \mid i \in I\}$ is called a frame for H, where the numbers A and B are called the lower and upper frame bounds of the frame, respectively. A frame is a tight frame if A and B can be chosen so that $A = B$, and is a normalized tight frame if $A = B = 1$.

Lemma 2.1 *$\{x_i \mid i \in I\}$ is a frame for H if and only if (1) holds for all $x \in \mathcal{M}$, where $\mathcal{M}$ is dense in H.*

The translation operator

and the inverse Fourier transform is

$$f^{\vee}(\xi) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\xi\cdot x$ denotes the standard inner product in $\mathbb{R}^n$. We also use the standard notation $\|f\|$ for the norm of $f \in L^2(\mathbb{R}^n)$, and $\langle f, g\rangle$ for the usual inner product of $f, g \in L^2(\mathbb{R}^n)$.

Throughout the paper, $\mathbb{Z}$ is the set of all integers, the space $\mathbb{T}^n$ will be identified with $[0,1]^n$, $GL_n(\mathbb{R})$ denotes the set of all non-singular $n\times n$ matrices with real entries and $\mathcal{D}$ is defined by

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^n) \mid \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \hat{f} \text{ has compact support in } \mathbb{R}^n\}.$$

It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^n)$. For $\forall C \in GL_n(\mathbb{R})$, let $C' := (C^t)^{-1}$, where C^t denotes the transpose of C.

3. Some Lemmas

In order to verify the results, we establish two lemmas in this section.

Lemma 3.1 *Let J be a countable indexing set, $\{f_j(x) \mid j \in J\} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. If* $\operatorname{ess\,sup}_{\xi} \frac{1}{|\det C|}\sum_{j\in J} |\hat{f}_j(\xi)|^2 < \infty$, *then for all $f \in \mathcal{D}$,*

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}^n} |\langle f, T_{Ck} f_j\rangle|^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{1}{|\det C|}\sum_{j\in J} |\hat{f}_j(\xi)|^2 d\xi + R(f), \tag{2}$$

where

$$R(f) = \sum_{j\in J}\sum_{m\neq 0} \frac{1}{|\det C|}\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi + C'm)\overline{\hat{f}(\xi + C'm)} d\xi. \tag{3}$$

The proof of Lemma 3.1 can be found in [12].

Lemma 3.2 *Let J be a countable indexing set, $\{f_j(x) \mid j \in J\} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. If* $\operatorname{ess\,sup}_{\xi} \frac{1}{|\det C|}\sum_{j\in J} |\hat{f}_j(\xi)|^2 < \infty$, *then for all $f \in \mathcal{D}$, the series in (3) is absolutely convergent.*

Proof. Notice that $|\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi + C'm)| \le \frac{1}{2}\left(|\hat{f}_j(\xi)|^2 + |\hat{f}_j(\xi + C'm)|^2\right)$, hence

$$\begin{aligned}
&\sum_{j\in J}\sum_{m\neq 0} \frac{1}{|\det C|}\left|\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi + C'm)\overline{\hat{f}(\xi + C'm)} d\xi\right| \\
\le\ & \frac{1}{2}\sum_{j\in J}\sum_{m\neq 0} \frac{1}{|\det C|}\int_{\mathbb{R}^n} |\hat{f}(\xi)\hat{f}(\xi + C'm)||\hat{f}_j(\xi)|^2 d\xi \\
&+ \frac{1}{2}\sum_{j\in J}\sum_{m\neq 0} \frac{1}{|\det C|}\int_{\mathbb{R}^n} |\hat{f}(\xi)\hat{f}(\xi + C'm)||\hat{f}_j(\xi + C'm)|^2 d\xi.
\end{aligned}$$

Obviously, it is enough to show that the series

$$\sum_{j\in J}\sum_{m\neq 0}\frac{1}{|\det C}$$

Theorem 4.1 *Let J be a countable indexing set, $\{f_j(x) \mid j \in J\} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. Set*

$$\beta_m = ess \sup_{\xi} \frac{1}{|\det C|} \left| \sum_{j \in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \right|, \quad \beta = \sum_{m \neq 0} (\beta_m \beta_{-m})^{\frac{1}{2}}.$$

If

$$A_1 = ess \inf_{\xi} \frac{1}{|\det C|} \sum_{j \in J} |\hat{f}_j(\xi)|^2 - \beta > 0 \tag{6}$$

and

$$B_1 = ess \sup_{\xi} \frac{1}{|\det C|} \sum_{j \in J} |\hat{f}_j(\xi)|^2 + \beta < +\infty, \tag{7}$$

then $\{T_{Ck} f_j(x) \mid k \in \mathbb{Z}^n, j \in J\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds A_1 and B_1.

Proof. By Lemma 2.1, it is sufficient to prove Theorem 4.1 for all $f \in \mathcal{D}$. By (7), Lemma 3.1 and Lemma 3.2, (2)holds, where

$$R(f) = \sum_{m \neq 0} \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{f}(\xi + C' m)} \sum_{j \in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) d\xi. \tag{8}$$

By the Cauchy-Schwarz inequality, the change of variables $\eta = \xi + C' m$ and (8), we get

$$\begin{aligned} |R(f)| &\leq \sum_{m \neq 0} \frac{1}{|\det C|} \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left| \sum_{j \in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \right| d\xi \right)^{\frac{1}{2}} \times \\ &\quad \left(\int_{\mathbb{R}^n} |\hat{f}(\xi + C' m)|^2 \left| \sum_{j \in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \right| d\xi \right)^{\frac{1}{2}} \\ &= \sum_{m \neq 0} \frac{1}{|\det C|} \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left| \sum_{j \in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \right| d\xi \right)^{\frac{1}{2}} \times \\ &\quad \left(\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 \left| \sum_{j \in J} \overline{\hat{f}_j(\eta)} \hat{f}_j(\eta - C' m) \right| d\xi \right)^{\frac{1}{2}} \\ &\leq \sum_{m \neq 0} (\beta_m \beta_{-m})^{\frac{1}{2}} \|f\|^2 \\ &= \beta \|f\|^2. \end{aligned} \tag{9}$$

Consequently, by (2), (6), (7) and (9),

$$A_1 \|f\|^2 \leq \sum_{j \in J} \sum_{k \in \mathbb{Z}^n} |\langle f, T_{Ck} f_j \rangle|^2 \leq B_1 \|f\|^2, \quad \forall f \in \mathcal{D}.$$

Therefore, the proof is completed.

Remark 4.1 It is easy to see that frame bounds of above theorem are better than ones of theorem 4.1 in [12].

Remark 4.2 It is not difficult to show that $\sum_{m\neq 0} \Lambda_m = \sum_{m\neq 0} \Lambda_{-m}$. Set $\Lambda' = \sum_{m\neq 0} \Lambda_m$, then by (8) and the Cauchy-Schwarz inequality, we have

$$|R(f)| \le \sum_{m\neq 0} (\Lambda_m \Lambda_{-m})^{\frac{1}{2}} \|f\|^2 \le \Lambda' \|f\|^2.$$

Now, the second sufficient condition is stated.

Theorem 4.2 *Let J be a countable indexing set, $\{f_j(x) \mid j \in J\} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. If*

$$A_2 = ess \inf_{\xi} \frac{1}{|\det C|} \sum_{j\in J} |\hat{f}_j(\xi)|^2 - \Lambda' > 0$$

and

$$B_2 = ess \sup_{\xi} \frac{1}{|\det C|} \sum_{j\in J} |\hat{f}_j(\xi)|^2 + \Lambda' < +\infty,$$

then $\{T_{Ck} f_j(x) \mid k \in \mathbb{Z}^n, j \in J\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds A_2 and B_2.

Using another estimation technique, we able to give the third sufficient condition for the system $\{T_{Ck} f_\alpha(x) \mid k \in \mathbb{Z}^n, \alpha \in \Lambda\}$ to be a frame for $L^2(\mathbb{R}^n)$.

Theorem 4.3 *Let J be a countable indexing set, $\{f_j(x) \mid j \in J\} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. If*

$$A_3 = ess \inf_{\xi} \left[\frac{1}{|\det C|} \left(\sum_{j\in J} |\hat{f}_j(\xi)|^2 - \sum_{m\neq 0} \Big| \sum_{j\in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \Big| \right) \right] > 0, \tag{10}$$

and

$$B_3 = ess \sup_{\xi} \left[\frac{1}{|\det C|} \sum_{m\in\mathbb{Z}^n} \Big| \sum_{j\in J} \overline{\hat{f}_j(\xi)} \hat{f}_j(\xi + C' m) \Big| \right] < +\infty, \tag{11}$$

then $\{T_{Ck} f_j(x) \mid k \in \mathbb{Z}^n, j \in J\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds A_3 and B_3.

Proof. By Lemma 2.1, it is sufficient to prove Theorem 4.3 for all $f \in \mathcal{D}$. We need to estimate $R(f)$ in (8) . It deduces from the Cauchy-Schwarz inequality and the change of variables $\eta = \xi + C^t m$ that

$$
\begin{aligned}
|R(f)| &\leq \sum_{m\neq 0}\frac{1}{|\det C|}\left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\Big|\sum_{j\in J}\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi+C^t m)\Big|d\xi\right)^{\frac{1}{2}}\times\\
&\quad\left(\int_{\mathbb{R}^n}|\hat{f}(\xi+C^t m)|^2\Big|\sum_{j\in J}\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi+C^t m)\Big|d\xi\right)^{\frac{1}{2}}\\
&\leq \left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\sum_{m\neq 0}\frac{1}{|\det C|}\Big|\sum_{j\in J}\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi+C^t m)\Big|d\xi\right)^{\frac{1}{2}}\times \qquad (12)\\
&\quad\left(\int_{\mathbb{R}^n}|\hat{f}(\eta)|^2\sum_{m\neq 0}\frac{1}{|\det C|}\Big|\sum_{j\in J}\overline{\hat{f}_j(\eta)}\hat{f}_j(\eta-C^t m)\Big|d\xi\right)^{\frac{1}{2}}\\
&= \int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\sum_{m\neq 0}\frac{1}{|\det C|}\Big|\sum_{j\in J}\overline{\hat{f}_j(\xi)}\hat{f}_j(\xi+C^t m)\Big|d\xi.
\end{aligned}
$$

By (2), (10), (11) and (12), we have

$$A_3\|f\|^2 \leq \sum_{j\in J}\sum_{k\in\mathbb{Z}^n}|\langle f, T_{Ck}f_j\rangle|^2 \leq B_3\|f\|^2, \quad \forall f\in\mathcal{D}.$$

Therefore, the proof of Theorem 4.3 is completed.

Remark 4.3 Obviously, frame bounds of this theorem are better than ones of theorem 4.2, and are also better than theorem 4.2 in [12].

5. Applications of main results to Gabor system

In this section, we apply Theorems 4.1, 4.2 and 4.3 to Gabor system, and then obtain some new results of Gabor frames. Let $g^l(x) \in L^2(\mathbb{R}^n)$, $l = 1, \cdots, L$, L be a positive integer and $g^l_{m,k} = M_{Bm}T_{Ck}g^l$. The Gabor system, which generated by $g^l(x)$ ($l = 1, \cdots, L$), is defined by

$$\{g^l_{m,k}(x) \mid m, k \in \mathbb{Z}^n,\ l = 1, \cdots, L\},$$

where $B, C \in GL_n(\mathbb{R})$. If we change the order of the translation and modulation operators, we also have the system

$$\{T_{Ck}M_{Bm}g^l(x) \mid m, k \in \mathbb{Z}^n,\ l = 1, \cdots, L\}.$$

It is immediate to see that $\{g^l_{m,k}(x) \mid m, k \in \mathbb{Z}^n,\ l = 1, \cdots, L\}$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\{T_{Ck}M_{Bm}g^l(x) \mid m, k \in \mathbb{Z}^n,\ l = 1, \cdots, L\}$ is a frame for $L^2(\mathbb{R}^n)$, and the frame bounds is the same in the two cases. In particular, $\{T_{Ck}M_{Bm}g^l(x) \mid m,k \in \mathbb{Z}^n,\ l = 1,\cdots,L\}$ is shift-invariant. So, the main results can apply directly to the Gabor system.

Let J be defined by $J = \{(l,m) \mid m \in \mathbb{Z}^n,\ l = 1,\cdots,L\}$, and $\forall\gamma \in J$, $f_\gamma(x) = M_{Bm}g^l(x)$. Then the system $\{T_{Ck}f_\gamma(x) \mid k \in \mathbb{Z}^n, \gamma \in J\}$ is the system $\{T_{Ck}M_{Bm}g^l(x) \mid m,k \in \mathbb{Z}^n,\ l = 1,\cdots,L\}$. Notice that $\forall\gamma = (l,m) \in J$, $\hat{f}_\gamma(\xi) = \hat{g}^l(\xi - Bm)$, hence $\forall k \in \mathbb{Z}^n$,

$$\sum_{\gamma\in J} \overline{\hat{f}_\gamma(\xi)}\hat{f}_\gamma(\xi + C^l k) = \sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} \overline{\hat{g}^l(\xi - Bm)}\hat{g}^l(\xi - Bm + C^l k).$$

Therefore, using Theorem 4.1, Theorem 4.2 and Theorem 4.3, respectively, we obtain

Theorem 5.1 *Let $B, C \in GL_n(\mathbb{R})$, $g^l(x) \in L^2(\mathbb{R}^n)$, $l = 1,\cdots,L$ and L be a positive integer. If*

$$C_1 = ess\inf_{\xi} \frac{1}{|\det C|}\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} |\hat{g}^l(\xi - Bm)|^2 - \Delta_G > 0,$$

$$D_1 = ess\sup_{\xi} \frac{1}{|\det C|}\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} |\hat{g}^l(\xi - Bm)|^2 + \Delta_G < \infty,$$

then $\{g^l_{m,k}(x) \mid m,k \in \mathbb{Z}^n,\ l = 1,\cdots,L\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1 and D_1, where

$$\Delta_G = \sum_{k\neq 0} (\Delta_k^G \Delta_{-k}^G)^{\frac{1}{2}}, \quad \Delta_k^G = ess\sup_{\xi} \frac{1}{|\det C|}\left|\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} \overline{\hat{g}^l(\xi - Bm)}\hat{g}^l(\xi - Bm + C^l k)\right|.$$

Theorem 5.2 *Let $B, C \in GL_n(\mathbb{R})$, $g^l(x) \in L^2(\mathbb{R}^n)$, $l = 1,\cdots,L$ and L be a positive integer. If*

$$C_2 = ess\inf_{\xi} \frac{1}{|\det C|}\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} |\hat{g}^l(\xi - Bm)|^2 - \Delta'_G > 0,$$

$$D_2 = ess\sup_{\xi} \frac{1}{|\det C|}\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} |\hat{g}^l(\xi - Bm)|^2 + \Delta'_G < \infty,$$

then $\{g^l_{m,k}(x) \mid m,k \in \mathbb{Z}^n,\ l = 1,\cdots,L\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_2 and D_2, where

$$\Delta'_G = \sum_{k\neq 0} \Delta_k^G, \quad \Delta_k^G = ess\sup_{\xi} \frac{1}{|\det C|}\left|\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^n} \overline{\hat{g}^l(\xi - Bm)}\hat{g}^l(\xi - Bm + C^l k)\right|.$$

Remark 5.1 In one dimensional case, theorem 5.2 was given in [13].

Theorem 6.3 *Let $B, C \in GL_n(\mathbb{R})$, $g^l(x) \in L^2(\mathbb{R}^n)$, $l = 1, \cdots, L$ and L be a positive integer. If*

$$C_3 = \operatorname{ess}\inf_{\xi} \frac{1}{|\det C|} \left(\sum_{l=1}^{L} \sum_{m \in \mathbb{Z}^n} |\hat{g}^l(\xi - Bm)|^2 - \sum_{k \neq 0} \left| \sum_{l=1}^{L} \sum_{m \in \mathbb{Z}^n} \overline{\hat{g}^l(\xi - Bm)}\, \hat{g}^l(\xi - Bm + C'k) \right| \right) > 0,$$

$$D_3 = \operatorname{ess}\sup_{\xi} \frac{1}{|\det C|} \left(\sum_{k \in \mathbb{Z}^n} \left| \sum_{l=1}^{L} \sum_{m \in \mathbb{Z}^n} \overline{\hat{g}^l(\xi - Bm)}\, \hat{g}^l(\xi - Bm + C'k) \right| \right) < +\infty,$$

then $\{g^l_{m,k}(x) \mid m, k \in \mathbb{Z}^n,\ l = 1, \cdots, L\}$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_3 and D_3.

Remark 5.2 If $n = 1$ and $L = 1$, then theorem 5.3 is just theorem 8.4.4 in [9]. In addition, the upper bound of theorem 5.3 is superior to one of theorem 2.3 in [13].

Since $\langle f, T_{Ck} M_{Bm} g^l \rangle = \langle f^\vee, (T_{Ck} M_{Bm} g^l)^\vee \rangle = \langle f^\vee, T_{-Bm} M_{Ck} (g^l)^\vee \rangle$ by the Plancherel Theorem, we able to present similar results in the time domain. Them were omitted here.

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